This is a summary of the concepts of integral calculus.

Definite integrals

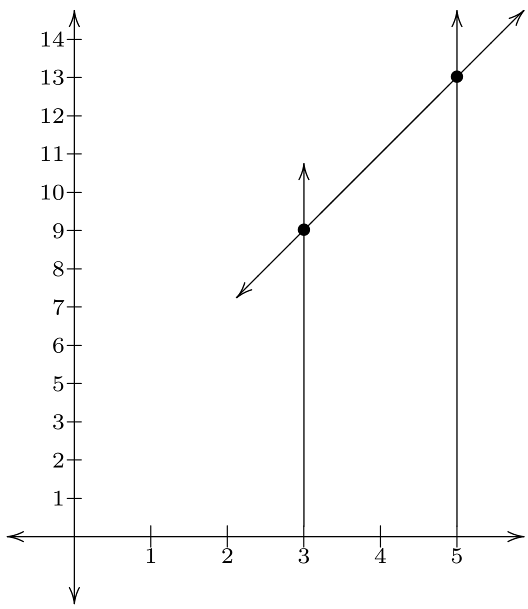
Just as the *differential* of a finitesimal (standard-size) quantity is an infinitesimal (infinitely small) change in that quantity, so the **definite integral** of an infinitesimal quantity is the sum of infinitely many values of that quantity, giving a finite result. If x and y are standard smooth quantities, then $y\,\mathrm{d}x$ is a typical infinitesimal smooth quantity. If we add this up from the point where $x = a$ to the point where $x = b$, then we get the **definite integral**

$$\int_{x=a}^{x=b} y\,\mathrm{d}x.$$

As long as the same variable x is used throughout, then it's safe to abbreviate this as

$$\int_a^b y\,\mathrm{d}x.$$

For example, $\int_3^5 (2t+3)\,\mathrm{d}t$ is the sum, as t varies smoothly from 3 to 5, of the product of $2t+3$ and the infinitesimal change in t at any given stage along the way. We can think of this product as giving the area of a rectangle whose height is $2t+3$ and whose width is $\mathrm{d}t$; if we line these rectangles up side by side, then they combine to give the area of a trapezoid:



We can find out the area of this trapezoid using geometry, since its width is $5 - 3 = 2$ and its height varies linearly from $2(3) + 3 = 9$ to $2(5) + 3 = 13$. Therefore,

$$\int_3^5 (2t+3)\,\mathrm{d}t = \frac{9+13}{2} \cdot 2 = 22.$$

Normally, you can't evaluate an integral by drawing a picture like this; I'll come back to how we can calculate it after a brief digression.

Antidifferentials

If $y\,\mathrm{d}x = \mathrm{d}u$, then u is an **antidifferential** of $y\,\mathrm{d}x$. However, u is *not the only* antidifferential of $y\,\mathrm{d}x$; if C is any constant, then $\mathrm{d}(u + C) = y\,\mathrm{d}x$ too, so $u + C$ is also an antidifferential of $y\,\mathrm{d}x$. However, for a continuously defined quantity, there is *no other* antidifferential of $y\,\mathrm{d}x$. Antidifferentials are denoted by '$\int$', so we have

$$\int \mathrm{d}u = u + C$$

by definition. An antidifferential is also called an **indefinite integral**.

For example,

$$\mathrm{d}(t^2 + 3t) = 2t\,\mathrm{d}t + 3\,\mathrm{d}t = (2t + 3)\,\mathrm{d}t,$$

so

$$\int (2t + 3)\,\mathrm{d}t = \int \mathrm{d}(t^2 + 3t) = t^2 + 3t + C.$$

As $2t + 3$ is the derivative of $t^2 + 3t$ with respect to t, we also say that $t^2 + 3t$ is an **antiderivative** of $2t + 3$ with respect to t.

To find antidifferentials (or antiderivatives), we must run the rules for differentials (and derivatives) backwards. This is often a subtle process, which I'll return to later.

The fundamental theorem of calculus

I now invite you to gaze, with awe and majesty, upon the **fundamental theorem of calculus**:

$$\int_a^b \mathrm{d}u = \Delta_a^b u.$$

Joking aside, this is important, and here's how you use it: If you want to evaluate a definite integral $\int_a^b y\,\mathrm{d}x$, then you should first figure out the indefinite integral $\int y\,\mathrm{d}x$. If the answer to this is u (or $u + C$), then this means that $y\,\mathrm{d}x = \mathrm{d}u$; that is, u is an antidifferential of $y\,\mathrm{d}x$. Therefore, $\int_a^b y\,\mathrm{d}x = \int_a^b \mathrm{d}u$, and the FTC tells us that this is equal to $\Delta_a^b u$. As this last expressions is simply a difference, you can figure it out using simple algebra.

For example, consider

$$\int_{t=3}^{t=5} (2t + 3)\,\mathrm{d}t.$$

In the last section, we saw that $\int (2t + 3)\,\mathrm{d}t = t^2 + 3t + C$; in other words, $(2t + 3)\,\mathrm{d}t = \mathrm{d}(t^2 + 3t)$. Therefore,

$$\int_3^5 (2t + 3)\,\mathrm{d}t = \Delta_3^5 (t^2 + 3t) = \big((5)^2 + 3(5)\big) - \big((3)^2 + 3(3)\big) = (40) - (18) = 22.$$

(Notice that this is the same answer as when I did this using geometry!)

This also explains why the same term 'integral' is used for both the definite integral (a sum of infinitely small quantities) and the indefinite integral (the antidifferential). They at first appear to be completely different concepts, but in reality they are closely related, through the fundamental theorem of calculus.

Integration techniques

This leaves us with one problem: how do we find indefinite integrals?

Each rule for differentiation gives us a rule for integration. In the table below, I have six important rules for differentiation (all of which you should know by now), together with a corresponding rule for integration:

$$\begin{aligned}
\mathrm{d}(u+v) &= \mathrm{d}u + \mathrm{d}v, & \int (y+z)\,\mathrm{d}x &= \int y\,\mathrm{d}x + \int z\,\mathrm{d}x;\\
\mathrm{d}(ku) &= k\,\mathrm{d}u, & \int ky\,\mathrm{d}x &= k\int y\,\mathrm{d}x;\\
\mathrm{d}(uv) &= v\,\mathrm{d}u + u\,\mathrm{d}v, & \int u\,\mathrm{d}v &= uv - \int v\,\mathrm{d}u;\\
\mathrm{d}(u^n) &= nu^{n-1}\,Du, & \int u^m\,\mathrm{d}u &= \frac{1}{m+1}u^{m+1} + C;\\
\mathrm{d}(\mathrm{e}^u) &= \mathrm{e}^u\,\mathrm{d}u, & \int \mathrm{e}^u\,\mathrm{d}u &= \mathrm{e}^u + C;\\
\mathrm{d}(\ln|u|) &= \frac{1}{u}\,\mathrm{d}u, & \int \frac{1}{u}\,\mathrm{d}u &= \ln|u| + C.
\end{aligned}$$

Using these rules, you can solve all of the problems in the book through Section 4.6.

For example, to find $\int (2x+3)\,\mathrm{d}x$:

$$\int (2x+3)\,\mathrm{d}x = \int 2x\,\mathrm{d}x + \int 3\,\mathrm{d}x = 2\int x^1\,\mathrm{d}x + 3\int \mathrm{d}x = \frac{2}{2}x^2 + 3x + C = x^2 + 3x + C.$$

This is the same answer as we got before, but this time I didn't have to guess the answer and get lucky; I was able to actually calculate it.

For more complicated integrals, there are fancier techniques. Rather than learn all of these, you can program them into a computer. You can even go to `http://integrals.wolfram.com/` for a free Internet service that will do this for you!

Summary

To find the indefinite integral $\int y\,\mathrm{d}x$, you need to use integration techniques; your answer will still have the variable in it and should end with an arbitrary constant C. To find the definite integral $\int_a^b y\,\mathrm{d}x$, first find the indefinite integral and then take a difference; your answer will be a constant (without the arbitrary C).