This is a summary of the concepts of differential calculus, from the primary perspective of the differential.

Differences

If a variable changes from the value a to the value b, then the difference between these two values is $b - a$. More generally, if one variable quantity x changes from a to b, then another variable u may change as well, but usually between different values. Whatever the difference in those values is, that is the **difference** in u as x varies from a to b. This may be denoted

$$\Delta_{x=a}^{x=b} u,\ \Delta_a^b u,\ \Delta u,$$

depending on how explicit the notation needs to be.

For example, let u be $2x + 3$, and consider $\Delta_{x=4}^{x=5} u$. Calculate:

$$\Delta_{x=4}^{x=5} u = \Delta_4^5 (2x + 3) = [2(5) + 3] - [2(4) + 3] = 13 - 11 = 2.$$

In other words, as x varies from 4 to 5, u varies from 11 to 13, and the difference between these is 2.

Differentials

The idea behind a differential is that it is an *infinitely small* difference. There are various ways to make this idea logically precise, but we will not go into that in this applied course. (I will return to this at the end of the course, if there is time.) In place of the uppercase Greek letter 'Δ' for a standard-sized (finitesimal) change, we use the lowercase Latin letter 'd' for an infinitely small (infinitesimal) change. So if u varies smoothly, then $\mathrm{d}u$ is the **differential** of u, which more or less means $\Delta_a^b u$ when $b - a$ is infinitely small (but somehow not quite zero).

Although this is usually not an issue in applied situations, it's important that u be a *smoothly varying* quantity, also called a **smooth variable**. Exactly what this means is, again, something that can be made precise. But for now, think of it as meaning that, whenever the underlying varying reality changes by a small amount, the variable quantity u also changes by a small amount: no sudden jumps or infinitely fast change.

Differences and differentials of linear expressions

The following rules hold exactly for differences:

- $\Delta k = 0$ if k is constant;
- $\Delta(u + v) = \Delta u + \Delta v$;
- $\Delta(ku) = k\,\Delta u$ if k is constant.

These equations hold for finitesimal changes, so they also hold for infinitesimal changes:

- $\mathrm{d}k = 0$ if k is constant (the **Constant Rule**);
- $\mathrm{d}(u + v) = \mathrm{d}u + \mathrm{d}v$ (the **Sum Rule**);
- $\mathrm{d}(ku) = k\,\mathrm{d}u$ if k is constant (the **Multiple Rule**).

This allows us to calculate differentials of linear expressions.

For example:

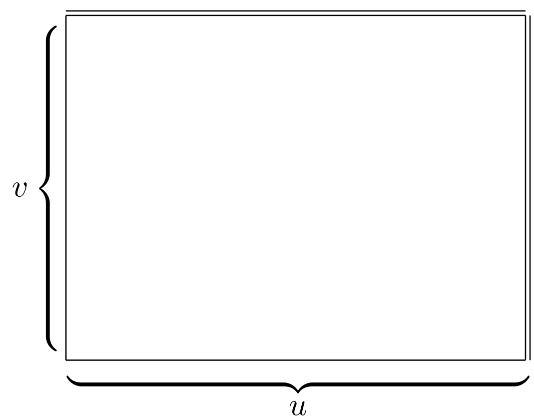
$$\begin{gathered}
\mathrm{d}(7x) = 7\,\mathrm{d}x; \\
\mathrm{d}(-5x) = -5\,\mathrm{d}x; \\
\mathrm{d}(x+2) = \mathrm{d}x + \mathrm{d}(2) = \mathrm{d}x + 0 = \mathrm{d}x; \\
\mathrm{d}(y-4) = \mathrm{d}y + \mathrm{d}(-4) = \mathrm{d}y + 0 = \mathrm{d}y; \\
\mathrm{d}(2t+3) = \mathrm{d}(2t) + \mathrm{d}(3) = 2\,\mathrm{d}t + 0 = 2\,\mathrm{d}t; \\
\mathrm{d}(7-x) = \mathrm{d}(-1x+7) = -1\,\mathrm{d}x + 0 = -\mathrm{d}x; \\
\mathrm{d}(2x+3y) = \mathrm{d}(2x) + \mathrm{d}(3y) = 2\,\mathrm{d}x + 3\,\mathrm{d}y; \\
\text{etc.}
\end{gathered}$$

Differentials of more complicated expressions

There is no simple rule for differences of expressions like x^2, or more generally products of variables such as uv. For differentials, however, we have the **Product Rule**:

$$\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v.$$

The reason for this may be seen by the following rectangle:



This rectangle has length u and height v, so its area is uv. However, both u and v are increasing, so the area is also increasing. (A similar picture could be drawn if one or both are decreasing instead.) The rectangle increases in two directions, upwards and to the right. Upwards, the increase is a strip of length u and height $\mathrm{d}v$, with an area of $u\,\mathrm{d}v$; to the right, the increase is a strip of length v and height $\mathrm{d}u$, with an area of $v\,\mathrm{d}u$. Therefore, the total change in the area, which is $\mathrm{d}(uv)$, is $u\,\mathrm{d}v + v\,\mathrm{d}u$, in accordance with the Product Rule. (It is precisely because we're looking only at *infinitesimal* changes that we can ignore the movement in the upper right corner of the rectangle.)

Using the Product Rule, we can derive rules to handle more general expressions. I will list all of the rules that we will need in other handouts; here I will show how some of them may be proved (assuming the previous rules).

Suppose that $v \neq 0$ and let $w = u/v$; then $vw = u$. Calculate:

$$\begin{aligned}
\mathrm{d}(vw) &= \mathrm{d}u; \\
w\,\mathrm{d}v + v\,\mathrm{d}w &= \mathrm{d}u; \\
v\,\mathrm{d}w &= \mathrm{d}u - w\,\mathrm{d}v; \\
\mathrm{d}w &= \frac{\mathrm{d}u - w\,\mathrm{d}v}{v}; \\
\mathrm{d}\left(\frac{u}{v}\right) &= \frac{\mathrm{d}u - \frac{u}{v}\,\mathrm{d}v}{v}; \\
\mathrm{d}\left(\frac{u}{v}\right) &= \frac{v\,\mathrm{d}u - u\,\mathrm{d}v}{v^2}.
\end{aligned}$$

The last line is the **Quotient Rule**.

Consider powers of u:

$$\begin{gathered}
\mathrm{d}(u^2) = \mathrm{d}(uu) = u\,\mathrm{d}u + u\,\mathrm{d}u = 2u\,\mathrm{d}u; \\
\mathrm{d}(u^3) = \mathrm{d}(u^2u) = u^2\,\mathrm{d}u + u\,\mathrm{d}(u^2) = u^2\,\mathrm{d}u + u(2u\,\mathrm{d}u) = u^2\,\mathrm{d}u + 2u^2\,\mathrm{d}u = 3u^2\,\mathrm{d}u; \\
\mathrm{d}(u^4) = \mathrm{d}(u^3u) = u^3\,\mathrm{d}u + u\,\mathrm{d}(u^3) = u^3\,\mathrm{d}u + u(3u^2\,\mathrm{d}u) = u^3\,\mathrm{d}u + 3u^3\,\mathrm{d}u = 4u^3\,\mathrm{d}u; \\
\text{etc.}
\end{gathered}$$

In general:

$$\mathrm{d}\big(u^k\big) = ku^{k-1}\,\mathrm{d}u$$

whenever k is a constant natural number.

Now consider negative powers. If k is a constant negative integer and $u \neq 0$, then $u^k u^{-k} = 1$. Calculate:

$$\begin{aligned}
\mathrm{d}\big(u^k u^{-k}\big) &= \mathrm{d}(1); \\
u^{-k}\,\mathrm{d}\big(u^k\big) + u^k\,\mathrm{d}\big(u^{-k}\big) &= 0; \\
u^{-k}\big(ku^{k-1}\,\mathrm{d}u\big) + u^k\,\mathrm{d}\big(u^{-k}\big) &= 0; \\
ku^{-1}\,\mathrm{d}u + u^k\,\mathrm{d}\big(u^{-k}\big) &= 0; \\
u^k\,\mathrm{d}\big(u^{-k}\big) &= -ku^{-1}\,\mathrm{d}u; \\
\mathrm{d}\big(u^{-k}\big) &= -ku^{-k-1}\,\mathrm{d}u.
\end{aligned}$$

Since also

$$\mathrm{d}(u^0) = \mathrm{d}(1) = 0 = 0u^{-1}\,\mathrm{d}u$$

if $u \neq 0$, the **Power Rule**

$$\mathrm{d}\big(u^k\big) = ku^{k-1}\,\mathrm{d}u$$

holds whenever k is a constant and the right-hand side is defined (at least when k is an integer, but we'll see shortly that it holds even when k is fractional).

Now consider roots. If k is a constant natural number, $u \neq 0$, and $\sqrt[k]{u}$ is defined as a real quantity, then $(\sqrt[k]{u})^k = u$. Calculate:

$$\begin{aligned}
\mathrm{d}\Big((\sqrt[k]{u})^k\Big) &= \mathrm{d}u; \\
k(\sqrt[k]{u})^{k-1}\,\mathrm{d}(\sqrt[k]{u}) &= \mathrm{d}u; \\
\frac{ku}{\sqrt[k]{u}}\,\mathrm{d}(\sqrt[k]{u}) &= \mathrm{d}u; \\
\mathrm{d}(\sqrt[k]{u}) &= \frac{\sqrt[k]{u}\,\mathrm{d}u}{ku}.
\end{aligned}$$

This is the **Root Rule**.

Another way to say this is that

$$\mathrm{d}\big(u^{1/k}\big) = \frac{1}{k}u^{\frac{1}{k}-1}\,\mathrm{d}u;$$

the Power Rule holds whenever k is a constant rational number. We may then argue that the Power Rule holds whenever k is *any* constant real number, because u^k is sandwiched between the various rational powers of u.

We have now derived all of the rules that we will need this month; the next handout will be a list of these rules together with some simplified special cases.

Strategy for calculating differentials

The general method for calculating the differential of an expression is to work from the outside in, reversing the order of operations to find out which rule to use.

For example, to differentiate $\sqrt{x^3y} + \frac{x}{y-3}$, we first use the rule for addition (the Sum Rule), since the final operation is addition. Then in the first summand, we use the rules for roots, then for multiplication, then for powers; while in the second summand, we use the rules for division, then for subtraction, then for constants. So:

$$\begin{aligned}
\mathrm{d}\left(\sqrt{x^3y} + \frac{x}{y-3}\right) &= \mathrm{d}\Big(\sqrt{x^3y}\Big) + \mathrm{d}\left(\frac{x}{y-3}\right) \\
&= \frac{\sqrt{(x^3y)}\,\mathrm{d}(x^3y)}{2(x^3y)} + \frac{(y-3)\,\mathrm{d}(x) - (x)\,\mathrm{d}(y-3)}{(y-3)^2} \\
&= \frac{\sqrt{x^3y}\big((y)\,\mathrm{d}(x^3) + (x^3)\,\mathrm{d}(y)\big)}{2x^3y} + \frac{(y-3)\,\mathrm{d}x - x\big(\mathrm{d}(y) - \mathrm{d}(3)\big)}{(y-3)^2} \\
&= \frac{\sqrt{x^3y}\Big(y\big(3(x)^2\,\mathrm{d}(x)\big) + x^3\,\mathrm{d}y\Big)}{2x^3y} + \frac{(y-3)\,\mathrm{d}x - x(\mathrm{d}y - 0)}{(y-3)^2} \\
&= \frac{\sqrt{x^3y}(3x^2y\,\mathrm{d}x + x^3\,\mathrm{d}y)}{2x^3y} + \frac{(y-3)\,\mathrm{d}x - x\,\mathrm{d}y}{(y-3)^2}.
\end{aligned}$$

The process is messy and can be tedious, but it should be straightforward.

In Calculus, it's usually considered OK to leave an expression as above. However, you could expand it out, simplify, and gather together the $\mathrm{d}x$ and $\mathrm{d}y$ terms:

$$\left(\frac{3\sqrt{x^3y}}{2x} + \frac{1}{y-3}\right)\mathrm{d}x + \left(\frac{\sqrt{x^3y}}{2y} - \frac{x}{(y-3)^2}\right)\mathrm{d}y.$$

Sometimes this will be useful. In any case, it's important that this can be done; every term in the final expression for a differential should have (as a factor) the differential of one (and only one) variable.

Derivatives

If u and v are smooth variables and $\mathrm{d}v \neq 0$, then v will change a little bit whenever u does. Another way to say this is that u cannot change unless v does, so we may view the change in u as induced by the change in v, as a result of the sensitivity of u to changes in v. This sensitivity is measured by the **derivative** of u with respect to v:

$$\frac{\mathrm{d}u}{\mathrm{d}v}.$$

Since 'derivative' is a rather generic term, this may also be called the **sensitivity** of u with respect to v or the **rate of change** of u with respect to v.

For example, if $x = 3t^2$, then calculate:

$$\begin{aligned}
\mathrm{d}x &= \mathrm{d}(3t^2);\\
\mathrm{d}x &= 3\,\mathrm{d}(t^2);\\
\mathrm{d}x &= 3(2t\,\mathrm{d}t);\\
\mathrm{d}x &= 6t\,\mathrm{d}t;\\
\frac{\mathrm{d}x}{\mathrm{d}t} &= 6t.
\end{aligned}$$

That is, the derivative of $3t^2$ with respect to t is $6t$; we sometimes write

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(3t^2) = 6t.$$

We can go on and find the derivative of $6t$ with respect to t:

$$\begin{aligned}
\mathrm{d}(\mathrm{d}x/\mathrm{d}t) &= \mathrm{d}(6t);\\
\mathrm{d}(\mathrm{d}x/\mathrm{d}t) &= 6\,\mathrm{d}t;\\
\frac{\mathrm{d}(\mathrm{d}x/\mathrm{d}t)}{\mathrm{d}t} &= 6.
\end{aligned}$$

The left-hand side here is often written '$\mathrm{d}^2x/\mathrm{d}t^2$', but this notation does not make literal sense the way that $\mathrm{d}x/\mathrm{d}t$ does. A better way to write the left-hand side of the equation above is as $(\mathrm{d}/\mathrm{d}t)^2x$; that is,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2(3t^2) = 6.$$

In words, the **second derivative** of $3t^2$ with respect to t is 6.

If we're not given a formula for a variable in terms of another, we can still try to find the derivative as long as we're given an equation relating them. For example, suppose that $x^2 + y^2 = 1$. Then calculate:

$$\begin{aligned}
\mathrm{d}(x^2 + y^2) &= \mathrm{d}(1); \\
\mathrm{d}(x^2) + \mathrm{d}(y^2) &= 0; \\
2x\,\mathrm{d}x + 2y\,\mathrm{d}y &= 0; \\
2y\,\mathrm{d}y &= -2x\,\mathrm{d}x; \\
\mathrm{d}y &= -\frac{x}{y}\,\mathrm{d}x; \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y}.
\end{aligned}$$

This is called an **implicit derivative**; although we've found the derivative of y with respect to x (assuming that $y \neq 0$), the expression for it involves both x and y and is not explicitly in terms of x alone.

Derivatives of functions

If we apply a function f to a variable x, then we may give a name to the result and say, for example, that

$$y = f(x).$$

If we had an explicit formula for f, then we could differentiate both sides of this equation and find that $\mathrm{d}y$ is some expression multiplied by $\mathrm{d}x$. Even without a formula for f, if we assume that f is a fixed **smooth function** (another concept that can be made precise), then $\mathrm{d}y$ is the product of $\mathrm{d}x$ and the result of applying some other function f'. That is,

$$\mathrm{d}\big(f(x)\big) = f'(x)\,\mathrm{d}x$$

if $f(x)$ depends only on x. This function f' is called the **derivative** of f, because $f'(x)$ is the derivative of $f(x)$ with respect to x. Then the derivate of f', denoted f'', is the **second derivative** of f, etc.

Notice that we take the derivative of one quantity with respect to another but we take the derivative of a function in an absolute sense; in symbols, $\mathrm{d}y/\mathrm{d}x$ is the derivative of y with respect to x, while f' is the derivative of f, period. Of course, the relationship between these ideas is that

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}.$$

For example, if $f(x) = x^2$, then calculate:

$$\begin{aligned}
\mathrm{d}f(x) &= \mathrm{d}(x^2); \\
\mathrm{d}f(x) &= 2x\,\mathrm{d}x; \\
\frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \frac{2x\,\mathrm{d}x}{\mathrm{d}x}; \\
f'(x) &= 2x.
\end{aligned}$$