Although limits are not the foundation in our approach to calculus, they are still an important concept. Roughly speaking, a limit describes what value one quantity is approaching as another quantity approaches some value in some direction.

Directions

Technically, a **direction** in some variable describes not only whether the variable is increasing or decreasing (that is its literal direction on a number line) but also if there is a limiting value that it approaches but does not reach. The basic directions that we study in this course take the following four forms, where x may be any variable and c may be any constant:

- as x increases without bound: $x \to \infty$;
- as x decreases without bound: $x \to -\infty$;
- as x increases towards c: $x \to c^-$;
- as x decreases towards c: $x \to c^+$.

Any two or more of these directions may be combined, but the only type of combined direction that we will need is this:

- as x approaches c: $x \to c$;

which is the combination of $x \to c^-$ and $x \to c^+$.

If D is any direction and u is any variable quantity, then we indicate the value to which u approaches as change occurs in the indicated direction as

$$\lim_D u.$$

We will examine the case when u approaches a real value L, as well as the case when u increases without bound or decreases without bound. In the first case, we say that the limit **converges**; in the second case, we say that the limit **diverges** to (positive or negative) infinity. Other types of behaviour are also possible, which are also kinds of divergence, but we consider those cases to be undefined. A limit as x approaches c exists (as one of the three kinds of results that we have defined) if and only if the limits as x increases and decreases towards c both exist *and* are the same result.

So in total, there are fifteen kinds of limits that we will consider, for the five kinds of directions (four basic and one combined) and the three kinds of answers:

$$\begin{array}{lll}
\lim\limits_{x\to\infty} u = L; & \lim\limits_{x\to\infty} u = \infty; & \lim\limits_{x\to\infty} u = -\infty; \\
\lim\limits_{x\to-\infty} u = L; & \lim\limits_{x\to-\infty} u = \infty; & \lim\limits_{x\to-\infty} u = -\infty; \\
\lim\limits_{x\to c^-} u = L; & \lim\limits_{x\to c^-} u = \infty; & \lim\limits_{x\to c^-} u = -\infty; \\
\lim\limits_{x\to c^+} u = L; & \lim\limits_{x\to c^+} u = \infty; & \lim\limits_{x\to c^+} u = -\infty; \\
\lim\limits_{x\to c} u = L; & \lim\limits_{x\to c} u = \infty; & \lim\limits_{x\to c} u = -\infty.
\end{array}$$

To see how to read these aloud, I'll consider the last one as an example; this says that the **limit** of u, as x approaches c, is negative infinity.

Evaluating limits

The first fact to know about limits is that the limit of the variable itself is already given by the direction:

$$\lim_{x \to \infty} x = \infty;$$
$$\lim_{x \to -\infty} x = -\infty;$$
$$\lim_{x \to c^-} x = c;$$
$$\lim_{x \to c^+} x = c.$$

Since the limit of x is c whenever x increases or decreases to c, the limit is still c whenever x approaches c:

$$\lim_{x \to c} x = c.$$

A similarly important principle is that the limit of a constant, in *any* direction, is that constant:

$$\lim_D k = k.$$

Of course, we rarely bother with limits as simple as these! However, we have the powerful principle that if an expression is built using only the basic operations of addition, subtraction, multiplication, and division, then the limit of the expression may be computed using these operations. Explicitly, each of these equations is true whenever the right-hand side is defined (so that in particular the left-hand side is automatically also defined):

$$\begin{aligned} \lim_D (u + v) &= (\lim_D u) + (\lim_D v); \\ \lim_D (u - v) &= (\lim_D u) - (\lim_D v); \\ \lim_D (uv) &= (\lim_D u)(\lim_D v); \\ \lim_D \left(\frac{u}{v}\right) &= \frac{\lim_D u}{\lim_D v}. \end{aligned}$$

Furthermore, the same principle applies to exponentiation and to taking roots, as long as the exponent is a constant:

$$\begin{aligned} \lim_D \left(u^k\right) &= (\lim_D u)^k; \\ \lim_D (\sqrt[k]{u}) &= \sqrt[k]{\lim_D u}. \end{aligned}$$

If you think about these rules in detail every time that you apply them, then you're working too hard. The point is simply that you can plug in the target c of the direction for the variable x and do ordinary arithmetic, as long as the result of this arithmetic is defined. In this way, we can evaluate most limits.

We can do even more limits if we extend arithmetic to the values $\pm\infty$ as follows, where a is (in general) any real number or $\pm\infty$ and k is a constant real number:

$$a + \infty = \infty \text{ if } a > -\infty;$$
$$a - \infty = -\infty \text{ if } a < \infty;$$
$$a \cdot \infty = \infty \text{ if } a > 0;$$
$$a \cdot \infty = -\infty \text{ if } a < 0;$$
$$a \div \infty = 0 \text{ if } -\infty < a < \infty;$$
$$\infty^k = \infty \text{ if } k > 0;$$
$$\infty^k = 0 \text{ if } k < 0;$$
$$\sqrt[k]{\infty} = \infty \text{ if } k > 0.$$

Technically, what these statements mean is:

- $\lim_D (u + v) = \infty$ if $\lim_D u > -\infty$ and $\lim_D v = \infty$;

and so on. But usually it's easier just to imagine calculating with infinity.

Finally, we can even divide by zero sometimes, *if* we are computing limits! Here I state the rules more carefully:

- $\lim_D (u/v) = \infty$ if $\lim_D u > 0$, $\lim_D v = 0$, and $v > 0$ in the direction D;
- $\lim_D (u/v) = -\infty$ if $\lim_D u > 0$, $\lim_D v = 0$, and $v < 0$ in the direction D;
- $\lim_D (u/v) = -\infty$ if $\lim_D u < 0$, $\lim_D v = 0$, and $v > 0$ in the direction D;
- $\lim_D (u/v) = \infty$ if $\lim_D u < 0$, $\lim_D v = 0$, and $v < 0$ in the direction D;
- $\lim_D (u/v)$ is undefined if $\lim_D u \neq 0$, $\lim_D v = 0$, and v takes both positive and negative values in the direction D.

In other words, if v has a consistent sign while it approaches 0, then the limit of u/v is $\pm\infty$, depending on how the sign of v compares to the sign of u. However, this tells us nothing if $\lim_D u = 0$ too; for that, we need the next idea.

L'Hôpital's Rule

We're left with the following **indeterminate forms**:

$$\infty - \infty,\ 0 \cdot \infty,\ \infty \div \infty,\ 0 \div 0.$$

Limits of these forms can usually be evaluated by using some more or less clever algebra to rewrite the expression in a more amenable way. However, another approach to the last two forms is given by **L'Hôpital's Rule**, assuming that the right-hand side is defined:

- $\lim_D \left(\frac{u}{v}\right) = \lim_D \left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)$ if $\lim_D u, \lim_D v = 0$ or $\lim_D u, \lim_D v = \pm\infty$.

Sometimes the other two indeterminate forms can also be rewritten as $0 \div 0$ or $\infty \div \infty$ using some more or less simple algebra.

This rule shows the connection between limits and differential calculus. In the official textbook, this connection is exploited in the other direction, to define derivatives (and so essentially differentials) using limits.