In Calculus, we study quantities that are *continuously* varying. That is, as a quantity y varies from (say) 2 to 5, it doesn't just pass through the values 3 and 4, but through all of the real numbers —all of the points on the number line— in between, such as $2\frac{1}{2}$, π, and 4.789.

In real life, we can never measure or fix the value of a such a quantity y exactly, down to the last decimal place; after all, there are infinitely many decimal places, but we can only do a finite amount of work. So, it is key to the study of real numbers that we can *approximate* them to any finite number of decimal places (among other ways).

Also in Calculus, we study how one quantity y varies along with another quantity x. This may be expressed by saying that y is a *function* of x; if f is the function, then $y = f(x)$. But in practice, we only know x and y *approximately*, so if we only use an approximate value of x, then $f(x)$ should still be an approximate value of y. For example, suppose that $f(x) = x^2$; if you know that x is approximately 2, then you know that $y = f(x)$ is approximately $2^2 = 4$.

This doesn't work with every function! For example, suppose that g is the piecewise-defined function

$$g(x) = \begin{cases} x+1 & \text{for } x < 2, \\ x+3 & \text{for } x \geq 2; \end{cases}$$

if you only know that x is approximately 2, then you really don't know if $g(x)$ is approximately $2 + 1 = 3$ or approximately $2 + 3 = 5$. Of course, if you knew that x is *exactly* 2, then you would know that $g(x)$ is $2 + 3 = 5$; but it's no good if you only know x approximately.

In these examples, we say that g has a **discontinuity** at 2, while f is **continuous** at 2. (In fact, f is continuous everywhere, while g is continuous everywhere except at 2.) So the idea is this:

> A function f is **continuous** at a real number c if, whenever $x \approx c$ (meaning that x is approximately equal to c), $f(x) \approx f(c)$.

So if you only know that $x \approx c$, then that's enough information to know $f(x)$ approximately (specifically, that $f(x) \approx f(c)$).

Actually, we should take care about where f is defined. Sometimes Calculus textbooks say that f has a discontinuity at c if $f(c)$ is undefined, and sometimes they don't; but in any case, f is not continuous there: f must be defined first in order to be continuous. On the other hand, if $f(x)$ is undefined, then we don't hold that against f; for example, we want to say that $f(x) = \sqrt{x}$ is continuous at 0, even though $f(x)$ is undefined (as a real number) whenever $x < 0$. So a more careful definition is this:

> A function f is **continuous** at a real number c if $f(c)$ is defined and, whenever $x \approx c$ and $f(x)$ is defined, $f(x) \approx f(c)$.

This is still not a completely rigorous definition, because it doesn't explain how close we need to be to say that one quantity is approximately equal to another. (Basically, the answer is this: as close as you need, and as close as you want.) But I will save that for a bit later. Already, this basic idea should be enough to allow you to judge continuity of a function from its graph.

To judge continuity of a function from a formula, know that any function is continuous (wherever it is defined) if it has a formula that uses only these operations: addition, subtraction, multiplication, division, absolute values, exponentation if the exponent is constant or the base is always positive, roots if the index is constant or the radicand is always positive, logarithms, trigonometric functions, and inverse trigonometric functions. These are pretty much all of the functions that you ever deal with!

So, the exceptions are much rarer: exponentiation where the exponent varies and the base can be zero or negative, roots where the index varies and the radicand can be zero or negative, and piecewise-defined functions. Of these, only piecewise-defined functions are likely to come up. These functions *can* be continuous, but only if the values agree on both sides whenever two pieces join. So for example, while

$$g(x) = \begin{cases} x+1 & \text{for } x < 2, \\ x+3 & \text{for } x \geq 2 \end{cases}$$

has a discontinuity at $x = 2$,

$$h(x) = \begin{cases} x+1 & \text{for } x < 2, \\ 5-x & \text{for } x \geq 2 \end{cases}$$

is continuous at $x = 2$ (and so everywhere), because $2 + 1 = 5 - 2$.

Returning to the meaning of continuity, how close of an approximation is close enough? The key to the answer is that a real number may be approximated as precisely as you wish, as long as you put enough work into it. So for f to be continuous at c, we should be able to demand that $f(x)$ and $f(c)$ be as close together as we like (as long as we still allow for a positive distance between them). But in order to achieve that result, it's fair in turn to demand that x be as close to c as necessary (again as long as we still allow the distance to be positive). The distance between two numbers is given by subtracting and taking the absolute value, so we need to be able to ensure that $|f(x) - f(c)|$ is as small as we want (but positive) by making $|x - c|$ as small as we need (but positive).

The traditional symbols for these small but positive distances are the Greek letters 'ϵ' (Epsilon) and 'δ' (Delta). For this reason, this is sometimes called the ϵ-δ (or epsilon-delta) definition; this method is also called *epsilontics*. So here is the rigorous definition:

> A function f is **continuous** at a real number c if $f(c)$ is defined and, for each positive number ϵ (no matter how small), there is some positive number δ (possibly quite small), such that whenever $|x - c| < \delta$ and $f(x)$ is defined, $|f(x) - f(c)| < \epsilon$.

This is fairly complicated, but you can view it as a game, involving a function f and a number c such that $f(c)$ exists.

- I challenge you with a positive number ϵ.
- You respond with a positive number δ.
- I reply with a value of x such that $|x - c| < \delta$ and $f(x)$ is defined.
- You win if $|f(x) - f(c)| < \epsilon$.

If you can win this game, no matter what choices I make, then f is continuous at c. On the other hand, if I can win no matter what choices you make, then f has a discontinuity at c.