Given any function $f$ and a number $c$ in the domain of $f$, the **difference quotient** of $f$ at $c$ is a function $\tilde{f}_c$, given by
$$\tilde{f}_c(h) = \frac{f(c+h) - f(c)}{h}.$$
Note that $\tilde{f}_c$ is not defined at 0. (In general, it's defined at any value $h$ such that $h \neq 0$ and $f$ is defined at $c+h$.) The **derivative** of $f$ at $c$ is the limit of $\tilde{f}_c$ approaching 0:
$$f'(c) = \lim_{h \to 0} \tilde{f}_c(h) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$
(When this exists, we say that $f$ is **differentiable** at $c$.) This is the usual definition, except that people usually don't bother to give a name to $\tilde{f}_c$.

Because limits are closely related to continuity, it's possible to give a definition of the derivative based on continuity. We extend the definition of $\tilde{f}_c$ like this:
$$\tilde{f}_c(h) = \begin{cases} \dfrac{f(c+h) - f(c)}{h} & \text{for } h \neq 0, \\ f'(c) & \text{for } h = 0. \end{cases}$$
If there exists a number $f'(c)$ that makes this function continuous at 0, then that number is the derivative of $f$ at $c$; if there isn't, then this derivative doesn't exist and $f$ is not differentiable at $c$. As it is, this is just the usual definition stated with different terminology. Now I'll do a little algebra on $\tilde{f}_c$: if $h \neq 0$ and $f$ is defined at $c+h$, then
$$\begin{aligned} \tilde{f}_c(h) &= \frac{f(c+h) - f(c)}{h}, \\ h\,\tilde{f}_c(h) &= f(c+h) - f(c), \\ h\,\tilde{f}_c(h) + f(c) &= f(c+h), \\ f(c+h) &= f(c) + \tilde{f}_c(h)\,h; \end{aligned}$$
if $h = 0$, then this equation is still true as long as $\tilde{f}_c$ is defined at 0, since then it just says that $f(c) = f(c)$. So another way to define the derivative is to say that $f$ is differentiable at $c$ if there exists a function $\tilde{f}_c$ that is continuous at 0 and satisfies the last equation above (for all $h$ such that $f$ is defined at $c+h$), and then $f'(c) = \tilde{f}_c(0)$.

This is useful, because having the entire function $\tilde{f}_c$ can help with proving theorems about derivatives, and in fact the general strategy is to apply the equation for $f(c+h)$. For example, to prove that $fg$ is differentiable wherever $f$ and $g$ are, with
$$(fg)'(c) = f'(c)\,g(c) + f(c)\,g'(c),$$
I'll use $\tilde{f}_c$ and $\tilde{g}_c$ along with the limit definition of $(fg)'$:
$$\begin{aligned} (fg)'(c) &= \lim_{h \to 0} \frac{(fg)(c+h) - (fg)(c)}{h} = \lim_{h \to 0} \frac{f(c+h)\,g(c+h) - f(c)\,g(c)}{h} \\ &= \lim_{h \to 0} \frac{\big(f(c) + \tilde{f}_c(h)\,h\big)\big(g(c) + \tilde{g}_c(h)\,h\big) - f(c)\,g(c)}{h} \\ &= \lim_{h \to 0} \frac{f(c)\,g(c) + f(c)\,\tilde{g}_c(h)\,h + \tilde{f}_c(h)\,h\,g(c) + \tilde{f}_c(h)\,h\,\tilde{g}_c(h)\,h - f(c)\,g(c)}{h} \\ &= \lim_{h \to 0} \frac{\tilde{f}_c(h)\,g(c)\,h + f(c)\,\tilde{g}_c(h)\,h + \tilde{f}_c(h)\,\tilde{g}_c(h)\,h^2}{h} = \lim_{h \to 0} \big(\tilde{f}_c(h)\,g(c) + f(c)\,\tilde{g}_c(h) + \tilde{f}_c(h)\,\tilde{g}_c(h)\,h\big) \\ &= \tilde{f}_c(0)\,g(c) + f(c)\,\tilde{g}_c(0) + \tilde{f}_c(0)\,\tilde{g}_c(0)\,0 = f'(c)\,g(c) + f(c)\,g'(c) + f'(c)\,g'(c)\,0 \\ &= f'(c)\,g(c) + f(c)\,g'(c). \end{aligned}$$

(To evaluate the limit near the end, we need $\tilde{f}_c$ and $\tilde{g}_c$ to be continuous at 0.) This is not much simpler than the book's proof (although I used smaller steps), but it's a little more straightforward, without the step where you add and subtract something without clearly knowing why it will help.

Similarly, we can prove the Chain Rule right away: if $g$ is differentiable at $c$ and $f$ is differentiable at $g(c)$, then $f \circ g$ is differentiable at $c$ and

$$(f \circ g)'(c) = f'\big(g(c)\big)\, g'(c).$$

I'll prove this using $\tilde{g}_c$ and $\tilde{f}_{g(c)}$:

$$\begin{aligned}
(f \circ g)'(c) &= \lim_{h \to 0} \frac{(f \circ g)(c+h) - (f \circ g)(c)}{h} = \lim_{h \to 0} \frac{f\big(g(c+h)\big) - f\big(g(c)\big)}{h} \\
&= \lim_{h \to 0} \frac{f\big(g(c) + \tilde{g}_c(h)\, h\big) - f\big(g(c)\big)}{h} = \lim_{h \to 0} \frac{f\big(g(c)\big) + \tilde{f}_{g(c)}\big(\tilde{g}_c(h)\, h\big)\, \tilde{g}_c(h)\, h - f\big(g(c)\big)}{h} \\
&= \lim_{h \to 0} \frac{\tilde{f}_{g(c)}\big(\tilde{g}_c(h)\, h\big)\, \tilde{g}_c(h)\, h}{h} = \lim_{h \to 0} \Big(\tilde{f}_{g(c)}\big(\tilde{g}_c(h)\, h\big)\, \tilde{g}_c(h)\Big) \\
&= \tilde{f}_{g(c)}\big(\tilde{g}_c(0)\, 0\big)\, \tilde{g}_c(0) = \tilde{f}_{g(c)}\big(g'(c)\, 0\big)\, g'(c) = \tilde{f}_{g(c)}(0)\, g'(c) \\
&= f'\big(g(c)\big)\, g'(c).
\end{aligned}$$

Now the proof is not only straightforward (or as straightforward as something so abstract can be), but also it can be done immediately and rigorously without postponing things until the end of the chapter.

This definition of derivative will be handy for some other proofs later on, such as for the Mean Value Theorem.

**Appendix: All of the theorems**

Here are complete statements of all of the basic rules of differentiation in the notation of derivatives of functions:

- Constant Rule: If $f$ is a constant function, then $f$ is differentiable at any number $c$ and $f'(c) = 0$.
- Sum Rule: If $f$ and $g$ are differentiable at $c$, then so is $f + g$ and
$$(f + g)'(c) = f'(c) + g'(c).$$
- Difference Rule: If $f$ and $g$ are differentiable at $c$, then so is $f - g$ and
$$(f - g)'(c) = f'(c) - g'(c).$$
- Multiple Rule: If $f$ is differentiable at $c$ and $k$ is a constant, then $kf$ is also differentiable at $c$ and
$$(kf)'(c) = k\, f'(c).$$
- Product Rule: If $f$ and $g$ are differentiable at $c$, then so is $fg$ and
$$(fg)'(c) = f'(c)\, g(c) + f(c)\, g'(c).$$
- Quotient Rule: If $f$ and $g$ are differentiable at $c$ and $g(c) \neq 0$, then $f/g$ is also differentiable at $c$ and
$$(f/g)'(c) = \frac{f'(c)\, g(c) - f(c)\, g'(c)}{g(c)^2}.$$
- Power Rule: If $f$ is the power function with exponent $n$ (so $f(x) = x^n$ for all $x$), then $f$ is differentiable at any number $c$ (unless $c = 0$ and $n < 1$) and $f'(c) = n\, c^{n-1}$.
- Chain Rule: If $g$ is differentiable at $c$ and $f$ is differentiable at $g(c)$, then $f \circ g$ is differentiable at $c$ and
$$(f \circ g)'(c) = f'\big(g(c)\big)\, g'(c).$$

Each of these can be proved by applying the definition of derivative in terms of limits for the derivative that we want to prove exists, the definition of derivative in terms of continuity for the derivatives that we already know exist (if any), and (except for the Power Rule) basic algebraic simplification.