Many calculations in calculus are easier to do using *differentials*. Furthermore, differentials and the related *differential forms* are often used in applications, especially (but not only) to physics. The official textbook covers differentials, but incompletely and only in one minor application. It then uses differentials again later (mostly in material for Calculus 2 and 3), but they are useful much earlier. So I will make heavy use of them.

## Variables

In Calculus, we study *variable* quantities, that is quantities whose values may vary (or change).

In Algebra, we often use the word 'variable' to refer to any quantity whose value we don't know, even if this value is fixed and never changes throughout the problem. In fact, the standard Algebra problem, solving an equation such as $2x + 3 = 5$, involves figuring out the value of the variable; so it had only one value all along, and we just had to figure out what it was. So if $x$ is a variable in an Algebra problem, and at some point we decide that the value of $x$ is 1, then this may well mean that $x$ is 1 throughout the entire problem. (That's not always the case in Algebra, but it often is.)

In Calculus, we take the word 'variable' more seriously. If $x$ is a variable in a Calculus problem, then $x$ might be 1 at some point, but it will probably be 6 at some other point in the problem. (And more often than not, it will take all of the values in between 1 and 6 along the way, such as $1\frac{1}{2}$, $\pi$, and 5.789.) Furthermore, if $x$ and $y$ are two variables that appear in the same problem, then the value of $y$ will usually change as the value of $x$ changes. Calculus is primarily about exactly this situation: figuring out *how* one quantity changes as another quantity changes.

In the simplest cases, it turns out that $y$ is a function of $x$; that is, there is a fixed function $f$ such that $y = f(x)$ remains true as $x$ and $y$ vary. (There are also situations where the function $f$ is changing as well, say $f(t) = t^2$ at one point and $f(t) = t^3$ at another point, but we're not going to deal with that now.) Calculus textbooks generally try to fit everything into this mould, but it doesn't always come out like this naturally. Often, you know that both $x$ and $y$ are changing, but it's not obvious that the value of $x$ at some point is enough information to figure out the value of $y$ at some point; yet when you write $y = f(x)$, you're assuming that this is enough information.

Most of the time, however, we can assume that there is some variable $t$, called the *independent variable*, such that every other variable in the problem is a function of $t$. That is, if $x$ and $y$ appear in the problem, then there are fixed functions $f$ and $g$ such that $x = f(t)$ and $y = g(t)$ throughout the problem. (Then $x$ and $y$ are called *dependent variables*, since their values depend on the values of $t$, through the functions $f$ and $g$.) But this variable $t$ might not show up directly! Calculus books will usually tell you (especially in word problems) that it's necessary to pick an independent variable, but it's enough to visualize the range of variation in the problem, and you can treat all of the variables on an equal footing. All the same, for the sake of formal definitions, we will assume that there is an independent variable $t$ and that every other variable is a function of it, even though in practice we don't have to identify it. (Of course, you don't have to call the independent variable '$t$', but I usually will.)

If we're not going to refer directly to $t$, then we're not going to refer directly to $f$ and $g$ either, so we need some way to refer to the values of these functions without referring to the functions themselves. Here is how we do it formally:

If $u = f(t)$, then $u|_{t=c} = f(c)$.

More generally, if $P$ is some statement that is only true once, then $P$ is equivalent to the statement $t = c$ for some value of $c$, so we can make sense of $u|_P$. Even if $P$ is a statement that might or might not only be true once, as long as every possible value of $u|_P$ is the same, then we can still make sense of $u|_P$. Finally, even if there are different possible values of $u|_P$, then the value of $u|_P$ still varies, but at least it doesn't vary as much as $u$ itself, since there are now fewer possibilities.

This all sounds very abstract (because it is), but the concrete application is straightforward; here are some examples:

$$\begin{aligned} x|_{x=5} &= 5, \\ (2x+3)|_{x=4} &= 2(4)+3 = 11, \\ (2x+3y)|_{\substack{x=4,\\ y=5}} &= 2(4)+3(5) = 23. \end{aligned}$$

Taking the last of these for example, there is no need to think about what $t$ is when $x = 4$ and $y = 5$; it's enough that no matter what $t$ may be, if $x = 4$ and $y = 5$, then $u = 2x + 3y$ is definitely $2(4) + 3(5) = 23$. So all that you have to do in practice is to plug and chug. Sometimes (generally only in the middle of a problem or in something theoretical) you can't work out the value completely; for example,

$$(2x+3y)|_{x=4} = 2(4) + 3(y|_{x=4}) = 8 + 3y|_{x=4}.$$

If we don't know anything more about the relationship between $x$ and $y$, then we don't know the value of $y$ when $x = 4$, so this is all that we can say in this example, but at least we were able to work out part of it.

### Notation and terminology

If $x$ is a variable, then $\mathrm{d}x$ is the **differential** of $x$. You can think of $\mathrm{d}x$ as indicating an infinitely small (infinitesimal) change in the value of $x$, or (better) the amount by which $x$ changes when an infinitesimal change is made (an infinitely small change in the value of the independent variable $t$). A precise definition is at the end of these notes, but you will *not* be tested directly on that; what you need to know is how to *use* differentials.

Note that $\mathrm{d}x$ is *not* d times $x$, and $\mathrm{d}x$ is also *not* exactly a function of $x$. Rather, $x$ (being a *variable* quantity) should itself be a function of some other quantity $t$, and $\mathrm{d}x$ is also a function of a sort; so d is an *operator*: something that turns one function into another function. However, an expression like $u\,\mathrm{d}x$ does involve multiplication: it is $u$ times the differential of $x$.

We often divide one differential by another; for example, $\mathrm{d}y/\mathrm{d}x$ is the result of dividing the differential of $y$ by the differential of $x$. The textbook introduces this notation early to stand for the *derivative* of $y$ with respect to $x$, and indeed it is that; but what the book doesn't tell you is that $\mathrm{d}y/\mathrm{d}x$ literally is $\mathrm{d}y$ divided by $\mathrm{d}x$. Unfortunately, $\mathrm{d}^2y/\mathrm{d}x^2$, the second derivative, is *not* literally $\mathrm{d}^2y = \mathrm{d}(\mathrm{d}y)$ divided by $\mathrm{d}x^2 = (\mathrm{d}x)^2$; for this reason, I prefer the notation $(\mathrm{d}/\mathrm{d}x)^2y$, meaning $(\mathrm{d}/\mathrm{d}x)(\mathrm{d}/\mathrm{d}x)y = (\mathrm{d}/\mathrm{d}x)(\mathrm{d}y/\mathrm{d}x) = \mathrm{d}(\mathrm{d}y/\mathrm{d}x)/\mathrm{d}x$.

### The Chain Rule

The most important fact about differentials is this: If $f$ is a differentiable function, then

$$\mathrm{d}\big(f(u)\big) = f'(u)\,\mathrm{d}u.$$

That is, the differential of $f(u)$ equals $f'(u)$ times the differential of $u$, where $f'$ is the derivative of $f$ (as a function). This fact not only shows the relationship between differentials and derivatives, but also (because $u$ could be any quantity) it encapsulates the **Chain Rule** in differential form. The Chain Rule is an important principle in calculus, which is often difficult to learn how to use; but with differentials it is easy.

In particular, if $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\big(f(x)\big)}{\mathrm{d}x} = \frac{f'(x)\,\mathrm{d}x}{\mathrm{d}x} = f'(x),$$

so $\mathrm{d}y$ divided by $\mathrm{d}x$ really is the derivative.

For example, suppose that you have discovered (say from the definition as a limit) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. Then this fact can be expressed in differential form:

$$\mathrm{d}(x^2) = \mathrm{d}\big(f(x)\big) = f'(x)\,\mathrm{d}x = 2x\,\mathrm{d}x. \qquad (*)$$

Conversely, if (by performing a calculation with differentials) you discover the equation (*) above, then you know the derivative of $f$ as well:

$$f'(x) = \frac{\mathrm{d}\big(f(x)\big)}{\mathrm{d}x} = \frac{\mathrm{d}(x^2)}{\mathrm{d}x} = \frac{2x\,\mathrm{d}x}{\mathrm{d}x} = 2x.$$

Whichever of these facts you discover first, once you know them, you know something even more general:

$$\mathrm{d}(u^2) = 2u\,\mathrm{d}u.$$

(The power to derive this from equation (*) is the Chain Rule.) The value of this is that $u$ can be any expression whatsoever; for example, if $u = x^2$ again, then

$$\mathrm{d}(x^4) = \mathrm{d}\big((x^2)^2\big) = 2(x^2)\,\mathrm{d}(x^2) = 2x^2(2x\,\mathrm{d}x) = 4x^3\,\mathrm{d}x.$$

So now you have learnt a new derivative, without having to calculate it from scratch.

**Rules of differentiation**

Every theorem about derivatives of functions may also be expressed as a theorem about differentials. Here are the most common rules:

- The Constant Rule: $\mathrm{d}(K) = 0$ if $K$ is constant.
- The Sum Rule: $\mathrm{d}(u + v) = \mathrm{d}u + \mathrm{d}v$.
- The Translate Rule: $\mathrm{d}(u + C) = \mathrm{d}u$ if $C$ is constant.
- The Difference Rule: $\mathrm{d}(u - v) = \mathrm{d}u - \mathrm{d}v$.
- The Product Rule: $\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v$.
- The Multiple Rule: $\mathrm{d}(ku) = k\,\mathrm{d}u$ if $k$ is constant.
- The Quotient Rule: $\mathrm{d}\left(\frac{u}{v}\right) = \frac{v\,\mathrm{d}u - u\,\mathrm{d}v}{v^2}$.
- The Power Rule: $\mathrm{d}(u^n) = nu^{n-1}\,\mathrm{d}u$ if $n$ is constant.
- The Root Rule: $\mathrm{d}(\sqrt[m]{u}) = \frac{\sqrt[m]{u}\,\mathrm{d}u}{mu}$ if $m$ is constant.

Of these, only the Constant Rule, the Sum Rule, the Product Rule, and the Power Rule are absolutely necessary, since every other expression built out of the operations in the rules above can be built out of the operations in these four rules. However, it is often handy to use all of these rules, even the Root Rule (which is not in the textbook). It is up to you how many of these rules to learn.

In addition, every time that you learn the derivative of a new function, you learn a new rule for differentials, by applying the Chain Rule to that function. We have already seen an example of this: applying the Chain Rule to the function $f(x) = x^2$ gives the special case of the Power Rule for $n = 2$. Here are a few other functions whose derivatives you will learn, expressed as rules for differentials:

- $\mathrm{d}(\mathrm{e}^u) = \mathrm{e}^u\,\mathrm{d}u$.
- $\mathrm{d}(\ln u) = \frac{\mathrm{d}u}{u}$.
- $\mathrm{d}(\sin u) = \cos u\,\mathrm{d}u$.
- $\mathrm{d}(\cos u) = -\sin u\,\mathrm{d}u$.
- $\mathrm{d}(\arctan u) = \frac{\mathrm{d}u}{u^2 + 1}$.

And more!

Notice that every one of these rules turns the differential on the left into a sum of terms (possibly only one term, or none in the case of the Constant Rule), each of which is an ordinary expression multiplied by a differential (or something algebraically equivalent to this). An expression like this is called a **differential form** (although actually there are more general sorts of differential forms). If, when you are calculating the differential of an expression, your result at any stage is *not* like this, then you have made a mistake!

**Using differentials**

The main technique for using differentials is simply to take the differential of both sides of an equation. However, you may only do this to an equation that holds *generally*, but *not* to an equation that holds only for *particular* values of the variables. (Ultimately, this is because d is an operator, not a function, so it must be applied to entire functions, not only to particular values of those functions.)

The simplest case is an equation such as $y = \mathrm{e}^{x^2}$, when we want the derivative of $y$ with respect to $x$. So:

$$\begin{aligned} y &= \mathrm{e}^{x^2}; \\ \mathrm{d}y &= \mathrm{d}\left(\mathrm{e}^{x^2}\right) = \mathrm{e}^{x^2}\,\mathrm{d}(x^2) = \mathrm{e}^{x^2} \cdot 2x\,\mathrm{d}x = 2x\mathrm{e}^{x^2}\,\mathrm{d}x; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\mathrm{e}^{x^2}. \end{aligned}$$

Now we have the derivative. If we want the second derivative, then we do this again:

$$\begin{aligned} \mathrm{d}y/\mathrm{d}x &= 2x\mathrm{e}^{x^2}; \\ \mathrm{d}(\mathrm{d}y/\mathrm{d}x) &= \mathrm{d}\left(2x\mathrm{e}^{x^2}\right) = \mathrm{e}^{x^2}\,\mathrm{d}(2x) + 2x\,\mathrm{d}\left(\mathrm{e}^{x^2}\right) \\ &= \mathrm{e}^{x^2} \cdot 2\,\mathrm{d}x + 2x \cdot 2x\mathrm{e}^{x^2}\,\mathrm{d}x = \left(2\mathrm{e}^{x^2} + 4x^2\mathrm{e}^{x^2}\right)\mathrm{d}x; \\ (\mathrm{d}/\mathrm{d}x)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} &= 2\mathrm{e}^{x^2} + 4x^2\mathrm{e}^{x^2}. \end{aligned}$$

Now we have the second derivative (also written $\mathrm{d}^2y/\mathrm{d}x^2$).

The previous example began with an equation solved for $y$. But we don't need this; suppose instead that we have $y^5 + x^2 = x^5 + y$ (which *cannot* be solved for either variable using the usual algebraic operations of addition, subtraction, multiplication, division, powers, and roots). Undaunted, we forge ahead anyway:

$$\begin{aligned} y^5 + x^2 &= x^5 + y; \\ \mathrm{d}(y^5 + x^2) &= \mathrm{d}(x^5 + y); \\ \mathrm{d}(y^5) + \mathrm{d}(x^2) &= \mathrm{d}(x^5) + \mathrm{d}y; \\ 5y^{5-1}\,\mathrm{d}y + 2x^{2-1}\,\mathrm{d}x &= 5x^{5-1}\,\mathrm{d}x + \mathrm{d}y; \\ 5y^4\,\mathrm{d}y - \mathrm{d}y &= 5x^4\,\mathrm{d}x - 2x\,\mathrm{d}x; \\ (5y^4 - 1)\,\mathrm{d}y &= (5x^4 - 2x)\,\mathrm{d}x; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{5x^4 - 2x}{5y^4 - 1}. \end{aligned}$$

This process is called **implicit differentiation**.

The second derivative is a little more straightforward at first (or it would be if we didn't have to use the Quotient Rule), but there is a twist at the end:

$$
\begin{aligned}
\mathrm{d}y/\mathrm{d}x &= \frac{5x^4 - 2x}{5y^4 - 1}; \\
\mathrm{d}(\mathrm{d}y/\mathrm{d}x) &= \mathrm{d}\Big(\frac{5x^4 - 2x}{5y^4 - 1}\Big) = \frac{(5y^4 - 1)\,\mathrm{d}(5x^4 - 2x) - (5x^4 - 2x)\,\mathrm{d}(5y^4 - 1)}{(5y^4 - 1)^2} \\
&= \frac{(5y^4 - 1)(20x^3 - 2)\,\mathrm{d}x - (5x^4 - 2x)(20y^3)\,\mathrm{d}y}{(5y^4 - 1)^2} \\
&= \frac{20x^3 - 2}{5y^4 - 1}\,\mathrm{d}x - \frac{20y^3(5x^4 - 2x)}{(5y^4 - 1)^2}\,\mathrm{d}y; \\
(\mathrm{d}/\mathrm{d}x)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} &= \frac{20x^3 - 2}{5y^4 - 1} - \frac{20y^3(5x^4 - 2x)}{(5y^4 - 1)^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} \\
&= \frac{20x^3 - 2}{5y^4 - 1} - \frac{20y^3(5x^4 - 2x)}{(5y^4 - 1)^2}\,\frac{5x^4 - 2x}{5y^4 - 1}
\end{aligned}
$$

(which could be simplified further). Notice that I substitute the known expression for $\mathrm{d}y/\mathrm{d}x$ in the last step.

Another handy application of differentials is the case where both quantities $x$ and $y$ may be expressed as functions of some other quantity $t$. (For the purposes of formal definitions, we always assume that this is possible, but now we're really going to use it.) If we start with the same equation as above, then this will give us an equation relating the derivatives with respect to $t$:

$$
\begin{aligned}
y^5 + x^2 &= x^5 + y; \\
\mathrm{d}(y^5 + x^2) &= \mathrm{d}(x^5 + y); \\
\mathrm{d}(y^5) + \mathrm{d}(x^2) &= \mathrm{d}(x^5) + \mathrm{d}y; \\
5y^{5-1}\,\mathrm{d}y + 2x^{2-1}\,\mathrm{d}x &= 5x^{5-1}\,\mathrm{d}x + \mathrm{d}y; \\
5y^4\frac{\mathrm{d}y}{\mathrm{d}t} + 2x\frac{\mathrm{d}x}{\mathrm{d}t} &= 5x^4\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}.
\end{aligned}
$$

If we have information about one or both of these derivatives, then this equation will often give us useful information to solve a problem. This situation is called **related rates**, since derivatives can be viewed as rates of change (especially derivatives with respect to time $t$, although the $t$ in the equation above doesn't have to stand for time).

When we get to integrals, differentials become so useful that even the book starts using them, but I'll save that for later.

### Appendix: Definitions and proofs

To formally define what differentials are and prove their properties, I'll make the same assumption that I made at the beginning of these notes, that there is an independent variable $t$ that every other variable is a function of. Then, I said that if $u = f(t)$, then $u|_{t=c} = f(c)$. Now I'll say that, if $u = f(t)$ and $f$ is a differentiable function, then

$$\mathrm{d}u|_{\substack{t=c,\\ \mathrm{d}t=h}} = f'(c)\,h.$$

More generally, if $u = f(t)$ and $v = g(t)$, then

$$(u\,\mathrm{d}v)|_{\substack{t=c,\\ \mathrm{d}t=h}} = f(c)\,g'(c)\,h.$$

Again, this is abstract, but the concrete application is straightforward; for example:

$$(2x\,\mathrm{d}x + 3\,\mathrm{d}x)\big|_{\substack{x=4,\\ \mathrm{d}x=0.05}} = 2(4)(0.05) + 3(0.05) = 0.55,$$
$$(2x\,\mathrm{d}x + 3y\,\mathrm{d}y)\big|_{\substack{x=4,y=5,\\ \mathrm{d}x=0.05,\mathrm{d}y=0.02}} = 2(4)(0.05) + 3(5)(0.02) = 0.7.$$

(I've put small numbers in for $\mathrm{d}x$ and $\mathrm{d}y$, because this is most often what comes up in practice, although for theoretical purposes it doesn't matter.) It's now more common to be given only partial information; for example:

$$(2x\,\mathrm{d}x + 3\,\mathrm{d}x)\big|_{x=4} = 2(4)\,\mathrm{d}x + 3\,\mathrm{d}x = 11\,\mathrm{d}x,$$
$$(2x\,\mathrm{d}x + 3y\,\mathrm{d}y)\big|_{\substack{x=4,\\ y=5}} = 2(4)\,\mathrm{d}x + 3(5)\,\mathrm{d}y = 8\,\mathrm{d}x + 15\,\mathrm{d}y.$$

Notice that you don't plug in the values of $x$ and $y$ inside the differential operator d; if you're not given values of $\mathrm{d}x$ and $\mathrm{d}y$, then those differentials must remain in the answer.

While expressions like the above come up occasionally, the main purpose of a precise definition is to prove theorems. (That's how we can be sure that the rules of Calculus will always work, at least when the definitions that prove them can be made to apply.) Earlier I gave a list of rules for differentials; we can prove these using the precise definition of differential and the known rules for derivatives of functions. For example, if $u = f(t)$ and $v = g(t)$, then $uv = f(t)\,g(t) = (fg)(t)$. Therefore,

$$\mathrm{d}(uv)\big|_{\substack{t=c,\\ \mathrm{d}t=h}} = (fg)'(c)\,h = \big(f'(c)\,g(c) + f(c)\,g'(c)\big)\,h = g(c)\,f'(c)\,h + f(c)\,g'(c)\,h = (v\,\mathrm{d}u + u\,\mathrm{d}v)\big|_{\substack{t=c,\\ \mathrm{d}t=h}}.$$

Here, I've used the formal definition of differential along with the Product Rule for derivatives of functions. The conclusion is that $\mathrm{d}(uv)$ and $v\,\mathrm{d}u + u\,\mathrm{d}v$ always evaluate to the same result, so

$$\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v,$$

which is the Product Rule for differentials. In the same way, all of the rules for differentials follow from rules for derivatives of functions.

The Chain Rule is an important special case, so I'll prove it too. If $u = g(t)$ and $f$ is any function, then $f(u) = f(g(t)) = (f \circ g)(t)$, so if $f$ is differentiable, then

$$\mathrm{d}\big(f(u)\big)\Big|_{\substack{t=c,\\ \mathrm{d}t=h}} = \mathrm{d}\big((f\circ g)(t)\big)\Big|_{\substack{t=c,\\ \mathrm{d}t=h}} = (f\circ g)'(c)\,h = f'\big(g(c)\big)\,g'(c)\,h = \big(f'(u)\,\mathrm{d}u\big)\Big|_{\substack{t=c,\\ \mathrm{d}t=h}}.$$

Again, I used the definition of differential and the Chain Rule for functions, and my conclusion is the Chain Rule for differentials:

$$\mathrm{d}\big(f(u)\big) = f'(u)\,\mathrm{d}u$$

whenever $f$ is a differentiable function.

It's not really essential to assume that there exists a *single* independent variable that every other variable is a function of, and I'll stop making that assumption in Calculus 3. Then the formal definition will become a little trickier, but all of the rules for differentials will continue to apply exactly as I stated them above.