There are four main operations considered in Calculus: limits, derivatives (or differentials), integrals (or antidifferentials), and sums of infinite series. (The last of these is not covered until next term.) Here we will look at the first one: limits.

The simplest type of limit is

$$\lim_{x \to c} f(x),$$

which may be read 'the limit, as x approaches c, of $f(x)$'. Here, x must be an explicit variable, while c is a real number, and f is a function. (Often, $f(x)$ will be given directly by a formula, and the function itself will have no name.) The value of this limit (if it exists) is some real number: as x gets closer to c, what value (if any) does $f(x)$ get closer to?

If f is continuous at c, then the answer is very simple: $f(x)$ gets closer to $f(c)$. More generally, if $f(x)$ gets closer to some number L, then the function g given by

$$g(x) = \begin{cases} f(x) & \text{for } x \neq c, \\ L & \text{for } x = c \end{cases}$$

is continuous at c. Formally, that is the definition of the limit: the value of the limit (if it exists) is whatever number L (if any) makes this function continuous at c.

Note that $f(c)$ is completely irrelevant here; we are defining $g(c)$ to be L, and we only care about $f(x)$ for x different from c. It doesn't matter whether f is even defined at c! However, f needs to be defined *sufficiently close* to c; that is, no matter how small a positive number δ is, there must be some number x, with $x \neq c$ but $|x - c| < \delta$, such that $f(x)$ exists. (When this is true, we say that c is a *limit point* of f.) This is just a technicality, not something that you actually need to check; but if this fails, then *every* number L will satisfy the definition! As long as c is a limit point of f, however, then there is at most one value of the limit.

Besides this, there are 14 other kinds of limits (so 15 in all). This is not as bad as it seems, because these come from combining 5 types of directions with 3 types of results. The 5 types of directions are

- $x \to c$ (as x approaches c),
- $x \to c^-$ (as x increases to c),
- $x \to c^+$ (as x decreases to c),
- $x \to \infty$ (as x increases without bound), and
- $x \to -\infty$ (as x decreases without bound).

The 3 types of results are

- L (a real number),
- ∞ (positive infinity), and
- $-\infty$ (negative infinity).

Actually, there are more types of limits than these, but these are the only types of directions or results that we consider in this course.

Rather than write down the definitions of all 14 remaining types of limits, I'll just write 4 examples for the 4 remaining types of directions and 2 examples for the 2 remaining types of results. Just to be clear, first recall from above that

$$\lim_{x \to c} f(x) = L$$

means that

$$\begin{cases} f(x) & \text{for } x \neq c, \\ L & \text{for } x = c \end{cases}$$

is continuous in x (meaning as a function of x) at c. Similarly,

$$\lim_{x \to c^-} f(x) = L$$

means that

$$\begin{cases} f(x) & \text{for } x < c, \\ L & \text{for } x = c \end{cases}$$

is continuous in x at c. (That is, we ignore what happens when $x > c$.) Next,

$$\lim_{x \to c^+} f(x) = L$$

means that

$$\begin{cases} f(x) & \text{for } x > c, \\ L & \text{for } x = c \end{cases}$$

is continuous in x at c. (That is, we ignore what happens when $x < c$.) Next,

$$\lim_{x \to \infty} f(x) = L$$

means that

$$\begin{cases} f(1/x) & \text{for } x > 0, \\ L & \text{for } x = 0 \end{cases}$$

is continuous in x at 0. (That is, we replace x with $1/x$ so that $x \to \infty$ becomes $x \to 0^+$.) Finally,

$$\lim_{x \to -\infty} f(x) = L$$

means that

$$\begin{cases} f(1/x) & \text{for } x < 0, \\ L & \text{for } x = 0 \end{cases}$$

is continuous in x at 0. (That is, we replace x with $1/x$ so that $x \to -\infty$ becomes $x \to 0^-$.) On the other end,

$$\lim_{x \to c} f(x) = \infty$$

means that

$$\begin{cases} 1/f(x) & \text{for } x \neq c, \\ 0 & \text{for } x = c \end{cases}$$

is continuous in x at c *and* $f(x)$ is positive for x sufficiently close to c. Finally,

$$\lim_{x \to c} f(x) = -\infty$$

means that

$$\begin{cases} 1/f(x) & \text{for } x \neq c, \\ 0 & \text{for } x = c \end{cases}$$

is continuous in x at c and $f(x)$ is negative for x sufficiently close to c.

The official textbook defines limits directly using epsilontics (which is very similar to the epsilontic definition of continuity but slightly more complicated), then defines continuity using limits; I have defined continuity using epsilontics and defined limits using continuity. Our definitions come in different orders, but they are equivalent. In any case, the most important method of calculating limits is this:

- If f is continuous at c, then $\lim_{x \to c} f(x) = f(c)$.

This fact makes *most* limits trivial to calculate; but it's the exceptions where all of the interesting stuff happens!