Recall from the October 19 notes that if $f$ is differentiable at $c$, then

$$f(c+h) = f(c) + \tilde{f}_c(h)\,h$$

for some function $\tilde{f}_c$ that's continuous at 0 (and then $\tilde{f}_c(0)$ is $f'(c)$). Since $\tilde{f}_c$ is continuous at 0, we can say that $\tilde{f}_c(h) \approx \tilde{f}_c(0)$ when $h \approx 0$, or in other words, $\tilde{f}_c(h) \approx f'(c)$ when $h \approx 0$. Putting this approximation in the equation above, we get

$$f(c+h) \approx f(c) + f'(c)\,h$$

when $h \approx 0$. Writing $x$ for $c+h$ (so that $h = x - c$), you can also put this as

$$f(x) \approx f(c) + f'(c)\,(x-c)$$

when $x \approx c$. While the left-hand side could be any function, the right-hand side is a linear function of $x$; this is the **linear approximation** to $f$ near $c$.

This is actually only the beginning of a whole series of approximations, each (typically) better than the one before it:

$f(x) \approx f(c)$, a constant, if $f$ is continuous at $c$;

$f(x) \approx f(c) + f'(c)\,(x-c)$, a linear function of $x$, if $f$ is differentiable at $c$;

$f(x) \approx f(c) + f'(c)\,(x-c) + \frac{1}{2}\,f''(c)\,(x-c)^2$, a quadratic function of $x$, if $f$ is twice differentiable at $c$;

$f(x) \approx f(c) + f'(c)\,(x-c) + \frac{1}{2}\,f''(c)\,(x-c)^2 + \frac{1}{6}\,f'''(c)\,(x-c)^3$, a cubic function of $x$, if $f$ is 3-times differentiable at $c$;

$\vdots$

This sequence of approximations is discussed in Section 9.8 of the textbook and covered in Calculus 2.

It's handy to describe linear approximation in terms of differentials and differences. While a differential represents an infinitesimal (infinitely small) change, a **difference** represents an appreciable or finitesimal (not infinitely small) change. As $x$ changes from $c$ to $c+h$, we say that the difference in $x$ is

$$\Delta x = (c+h) - c = h.$$

Meanwhile, if $y = f(x)$, then the difference in $y$ is

$$\Delta y = y|_{x=c+h} - y|_{x=c} = f(c+h) - f(c).$$

To be specific, we can write

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} = f(c+h) - f(c).$$

Then the linear approximation says

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} = f(c+h) - f(c) \approx f(c) + f'(c)\,h - f(c) = f'(c)\,h = \mathrm{d}y|_{\substack{x=c,\\ \mathrm{d}x=h}}.$$

So in the end, the linear approximation replaces differences with differentials. Although

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} \approx \mathrm{d}y|_{\substack{x=c,\\ \mathrm{d}x=h}}$$

is the proper way to put it, often one abbreviates this as

$$\Delta y \approx \mathrm{d}y.$$

(But really this only correct if we also have $\Delta x \approx \mathrm{d}x$, because that difference is also replaced by a differential in the approximation.) More generally, you can say that an equation involving differentials can be replaced by an approximate equation involving differences.