Applications of differentiation

Here are the last few applications of differentiation.

1 *Optimization*

Literally, **optimization** is making something the best, but we use it in math to mean **maximization**, which is making something the biggest. (You can imagine that the thing that you're maximizing is a numerical measure of how good the thing that you're optimizing is.) Essentially the same principles apply to **minimization**, which is making something the smallest. (And *pessimization* is making something the worst, although people don't use that term very much.) A generic term for making something the largest or smallest is **extremization**.

In theory, optimization is simply finding absolute extrema, which is most easily done for continuous functions on closed, bounded intervals. In that case, the maximum and minimum must both exist, by the Extreme Value Theorem, and each of them must occur at either the endpoint of the interval or where the derivative of the function is either zero or undefined. However, practical problems cannot always be modelled in this way, so we will need some more general techniques.

The key principle of applied optimization is this:

> A quantity u can only take a maximum or minimum value when its differential $\mathrm{d}u$ is zero or undefined.

If you write u as $f(x)$, where f is a fixed differentiable function and x is a quantity whose range of possible values you already understand (typically an interval), then $\mathrm{d}u = f'(x)\,\mathrm{d}x$. So u can only take an extreme value when its derivative (with respect to x) is zero or undefined or when you can no longer vary x however you please (which must occur at the extreme values of x and typically only then). This recreates the situation that I referred to above, finding the extreme values of a function defined on an interval. However, the principle that $\mathrm{d}u$ is zero or undefined applies even when u is not explicitly given as a function of anything else.

Be careful, because u might not have a maximum or minimum value! Assuming that u varies continuously (which it must if Calclulus is to be useful at all), then it must have a maximum and minimum value whenever the range of possibilities is *compact*; this means that if you pass continuously through the possibilities in any way, then you are always approaching some limiting possibility. (In terms of $u = f(x)$, this is the case when f is continuous and its domain, the range of possible values of x, is a closed and bounded interval.)

However, if the range of possibilities heads off to infinity in some way, or if there is an edge case that's not quite possible to reach, then you also have to take a limit to see what value u is approaching. (In terms of $u = f(x)$, if the interval is open or unbounded at either end, then there is a direction in which x could vary but in which there is no limiting value of x in the range of possibilities.) If any such limit is larger than every value that u actually reaches (which includes the possibility that a limit is ∞), then u has no maximum value; if any such limit is smaller than every value that u actually reaches (which includes the possibility that a limit is $-\infty$), then u has no minimum value.

So in the end, you look at these possibilities:

- when the derivative of u is zero or undefined,
- the extreme edge cases, and
- the limits approaching impossible limiting cases.

The largest value of u that you find in this way (regardless of whether this value is actually attained or is only approached in the limit) is called the *supremum* of u; similarly, the smallest value of u that you find is called the *infimum* of u. If u actually takes the value of its supremum, then that same value is also the *maximum* of u; but if u only approaches its supremum in a limit, then it has no maximum. Similarly, if u actually takes the value of its infimum, then that same value is also the *minimum* of u; but if u only approaches its infimum in a limit, then it has no minimum.

Here is a typical problem: The hypotenuse of a right triangle (maybe it's a ladder leaning against a wall) is fixed at 20 feet, but the other two sides of the triangle could be anything. Still, since it's a right triangle, we know that $x^2 + y^2 = 20^2$, where x and y are the lengths of legs of the triangle. Differentiating this, $2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$. Now suppose that we want to maximize or minimize the area of this triangle. Since it's a right triangle, the area is $A = \frac{1}{2}xy$, so $\mathrm{d}A = \frac{1}{2}y\,\mathrm{d}x + \frac{1}{2}x\,\mathrm{d}y$. If this is zero, then $\frac{1}{2}y\,\mathrm{d}x + \frac{1}{2}x\,\mathrm{d}y = 0$, to go along with the other equation $2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$.

The equations at this point will always be linear in the differentials, so think of this is a system of linear equations in the variables $\mathrm{d}x$ and $\mathrm{d}y$. There are various methods for solving systems of linear equations; I'll use the method of addition (aka elimination), but any other method should work just as well. So $\frac{1}{2}y\,\mathrm{d}x + \frac{1}{2}x\,\mathrm{d}y = 0$ becomes $2xy\,\mathrm{d}x + 2x^2\,\mathrm{d}y = 0$ (multiplying both sides by $4x$), while $2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$ becomes $2xy\,\mathrm{d}x + 2y^2\,\mathrm{d}y = 0$ (multiplying both sides by y). Subtracting these equations gives $(2x^2 - 2y^2)\,\mathrm{d}y = 0$, so either $\mathrm{d}y = 0$ or $x^2 = y^2$. Now, x and y can change freely as long as they're positive, but we have limiting cases: $x \to 0^+$ and $y \to 0^+$. Since $x^2 + y^2 = 400$, we see that $x^2 \to 400$ as $y \to 0$; since x is positive, this means that $x \to 20$ as $y \to 0$. Similarly, $y \to 20$ as $x \to 0$. In those cases, $A = \frac{1}{2}xy \to 0$. On the other hand, if $x^2 = y^2$, then $x = y$ (since they are both positive), so $x, y = 10\sqrt{2}$, since $x^2 + y^2 = 400$. In that case, $A = \frac{1}{2}xy = 100$.

So the largest area is 100 square feet, and while there is no smallest area, the area can get arbitrarily small with a limit of 0.

2 *Economic applications*

In word problems in economics or finance, a few quantities arise regularly, which you should know about.

- **Quantity** in this context has a specific meaning: the amount of a good or service made and/or sold in a given period of time. Quantity is thus measured in such units as pounds per week, items per year, or litres per hour. Quantity is variously denoted q or x.
- **Price** (or *unit price*) is the amount of money received for a given amount of goods or services. So price is measured in units such as dollars per pound or euros per item. Price is denoted p, a *lowercase* letter.
- **Revenue** is the amount of money received for goods or services in a given period of time. Revenue is measured in dollars per week, euros per year, etc. Revenue is denoted R, and we have this equation:

  $$R = qp.$$

  (Notice that the units make sense in this equation; amount over time, multiplied by money over amount, becomes money over time.)
- **Cost** is the amount of money that the business has to spend (in a given period of time) in order to produce and distribute their goods and services. (In this terminology, *cost* is completely different from *price*.) Like revenue, cost is measured in units of money over time.
- Finally, **profit** is the amount of money that the business makes and keeps in a given period of time. Unlike everything else here, it makes sense for profit to be negative. Profit is denoted P, an *uppercase* letter, and we have another equation:

  $$P = R - C.$$

In business, you generally want to maximize profit: make it not only positive but as large as possible. Even if you don't want to maximize profit as normally measured (because you care about something else besides money), economists typically try to calculate whatever else you care about and still say that you maximize profit (in a generalized sense).

For any of these quantities, we can discuss their average or marginal values. In this context, the **average** profit/cost/etc is the profit/cost/etc divided by the quantity:

$$\bar{P} = \frac{P}{q},\ \bar{C} = \frac{C}{q},\ \ldots.$$

(As you can see, a bar is used to indicate this ratio. Be careful; when we get to applications of integrals, this bar will be used to denote an average in a different way.) On the other hand, the **marginal** profit/cost/etc is the derivative of profit/cost/etc with respect to quantity:

$$P' = \frac{\mathrm{d}P}{\mathrm{d}q},\ C' = \frac{\mathrm{d}C}{\mathrm{d}q},\ \ldots.$$

(As you can see, a prime tick is used to indicate this derivative, which is safe in context because it always means the derivative respect to q. For a derivative with respect to time, which is also important in this context even though we aren't doing any examples of that in this class, a dot may be used instead.) Although the units for a marginal or average quantity are the same, they represent different things!

Finally, people also speak of the **marginal average** profit/cost/etc:

$$\begin{aligned} \bar{P}' &= \frac{\mathrm{d}(P/q)}{\mathrm{d}q} = \frac{qP' - P}{q^2} = \overline{P' - \bar{P}}, \\ \bar{C}' &= \frac{\mathrm{d}(C/q)}{\mathrm{d}q} = \frac{qC' - C}{q^2} = \overline{C' - \bar{C}}, \\ &\vdots \end{aligned}$$

The marginal profit is particularly important, since it must be zero when profit is maximized (as long as the maximum profit occurs when it is still possible to vary the quantity in any way desired); and since the marginal marginal profit (the second derivative of profit with respect to quantity) is typically negative, the profit really will be maximized when the marginal profit is zero. However, in the absence of information about the revenue, there is a rule of thumb that one should minimize the average cost instead, which means finding where the marginal average cost is zero.

3 *Newton's Method*

If you want to solve an equation $f(x) = 0$, then the Intermediate Value Theorem may give you a way to approximate the solution, but it is usually very inefficient. The Newton–Raphson Method (or simply Newton's Method) is usually much faster, although it doesn't always work. Here, you start with a guess x_0, then replace it with a (hopefully) better guess x_1, and so on. These guesses are computed in turn as follows:

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}, \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)}, \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)}, \\ &\vdots \end{aligned}$$

With any luck, none of these guesses will give $f'(x) = 0$ (which makes the next guess undefined) but eventually one will give $f(x) \approx 0$ to as close an approximation as one wants.

The Newton–Raphson Method is guaranteed to work under certain conditions given by the Newton–Kantorovich Theorem: If f is differentiable at a, $f(a)$ and $f'(a)$ are nonzero, f is twice differentiable strictly between a and $a - 2f(a)/f'(a)$, and

$$|f''(x)| \leq \frac{|f'(a)|^2}{2\,|f(a)|}$$

whenever x is strictly between a and $a - 2f(a)/f'(a)$, then Newton's Method will give a sequence of values that are strictly between a and $a - 2f(a)/f'(a)$, and that converge to a solution of $f(x) = 0$ in the sense that the limit $\lim_{n\to\infty} x_n$ exists and $f(\lim_{n\to\infty} x_n) = 0$.