Integrals

This is a summary of the concepts of integral calculus.

1 *Definite integrals*

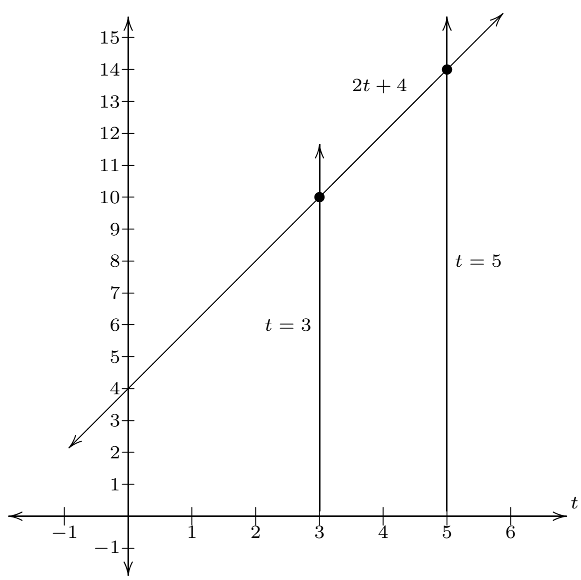
Just as the *differential* of a finite quantity is an infinitesimal (infinitely small) change in that quantity, so the **definite integral** of an infinitesimal quantity is the sum of infinitely many values of that quantity, giving a finite result. If x and y are standard quantities (neither infinitely large nor infintely small), then $y\,\mathrm{d}x$ is a typical infinitesimal quantity. (An expression like this is called a *differential form.*) If we add this up from the point where $x = a$ to the point where $x = b$, then we get the **definite integral**

$$\int_{x=a}^{b} y\,\mathrm{d}x.$$

As long as the same variable x is used throughout, then it's safe to abbreviate this as

$$\int_{a}^{b} y\,\mathrm{d}x.$$

For example, $\int_3^5 (2t+4)\,\mathrm{d}t$ is the sum, as t varies smoothly from 3 to 5, of the product of $2t+4$ and $\mathrm{d}t$ (the infinitesimal change in t) at each stage along the way. We can think of this product as giving the area of a rectangle whose height is $2t+4$ and whose width is $\mathrm{d}t$; if we line these rectangles up side by side, then they combine to give a trapezoid:



We can find out the area of this trapezoid using geometry, since its width is $5-3=2$ and its height varies linearly from $2(3)+4=10$ to $2(5)+4=14$. Therefore,

$$\int_3^5 (2t+4)\,\mathrm{d}t = \frac{10+14}{2}\cdot 2 = 24.$$

Normally, you can't evaluate an integral by drawing a picture like this; I'll come back to how we can calculate it after a brief digression.

2 *Antidifferentials*

If $\mathrm{d}u = y\,\mathrm{d}x$, then $y\,\mathrm{d}x$ is the *differential* of u, as you know. We also say that u is an **antidifferential** of $y\,\mathrm{d}x$. However, u is *not the only* antidifferential of $y\,\mathrm{d}x$; if C is any constant, then $\mathrm{d}(u + C) = y\,\mathrm{d}x$ too, so $u + C$ is also an antidifferential of $y\,\mathrm{d}x$. However, for a continuously defined quantity, there is *no other* antidifferential of $y\,\mathrm{d}x$. Even if there are gaps in the definition of the quantity, we can say that $u + C$ is an antidifferential of $\mathrm{d}u$ if and only if C is a *local* constant, meaning that it can change value only across a gap where u is undefined. (Ultimately, this is a consequence of the theorem that if the derivative of a function on an interval is always zero, then that function must be a constant; the relevant function here is the difference between the functions that give any two possible antidifferentials.)

Antidifferentials are denoted by '$\int$', so we have

$$\int \mathrm{d}u = u + C$$

by definition. (This looks similar to the notation for a definite integral, which makes sense reasons that will be explained below, but you can tell the difference because there are no bounds attached to the symbol.) For example,

$$\mathrm{d}(t^2 + 4t) = 2t\,\mathrm{d}t + 4\,\mathrm{d}t = (2t + 4)\,\mathrm{d}t,$$

so

$$\int (2t + 4)\,\mathrm{d}t = \int \mathrm{d}(t^2 + 4t) = t^2 + 4t + C.$$

As $2t + 4$ is the derivative of $t^2 + 4t$ with respect to t, we also say that $t^2 + 4t$ is an **antiderivative** of $2t + 4$ with respect to t. An antidifferential or antiderivative is also called an **indefinite integral**; so 'indefinite integral of $(t^2 + 4)\,\mathrm{d}t$' (antidifferential) and 'indefinite integral of $t^2 + 4$ with respect to t' (antiderivative) both mean $\int (t^2 + 4)\,\mathrm{d}t$.

To find antidifferentials (or antiderivatives), we must run the rules for differentials (and derivatives) backwards. This is often a subtle process, which I'll return to after a brief digression.

3 *The Fundamental Theorem of Calculus*

The **Fundamental Theorem of Calculus** relates definite and indefinite integrals. There are two parts:

1. $\mathrm{d}\Big(\int_{t=a}^{b} f(t)\,\mathrm{d}t\Big) = f(b)\,\mathrm{d}b - f(a)\,\mathrm{d}a$;
2. $\int_{t=a}^{b} \mathrm{d}f(t) = f(b) - f(a)$.

The first part applies whenever f is a continuous function (assuming that a and b are differentiable quantities); in particular, it claims that the integral exists and is differentiable. The second part applies whenever f is a differentiable function (assuming that t is a differentiable quantity); in particular, it claims that the integral exists.

Although both of these parts refer directly to definite integrals, indefinite integrals (antidifferentials) appear implicitly because of the presence of the differentials. Specifically, the first part claims that the definite integral that appears in it is an antidifferential of the differential form on its right-hand side, and the second part shows how to evaluate a definite integral of a differential form whose antidifferential is known.

If you want to express these without refering to the function f, then you can write them thus:

1. $\mathrm{d}\Big(\int_a^b \omega\Big) = \omega|_a^b$;
2. $\int_a^b \mathrm{d}u = u|_a^b$.

Here, I'm using ω to stand for an entire differential form (for which people often use Greek letters) and $u|_a^b$ is short for $u|_b - u|_a$. These basically say that d and $\int$ cancel as long as you move the bounds on the integral into bounds on a difference.

It's the second part of the theorem that we use the most. If you want to evaluate a definite integral $\int_a^b y\,\mathrm{d}x$, then you should first figure out the indefinite integral $\int y\,\mathrm{d}x$. If the answer to this is u (or more generally $u + C$), then this means that $y\,\mathrm{d}x = \mathrm{d}u$; that is, u is an antidifferential of $y\,\mathrm{d}x$. Therefore, $\int_{x=a}^{b} y\,\mathrm{d}x =$

$\int_{x=a}^{b} \mathrm{d}u$, and the FTC tells us that this is equal to $u|_{x=a}^{b}$. As this last expression is simply a difference, you can figure it out using simple algebra.

For example, consider

$$\int_{t=3}^{5} (2t+4)\,\mathrm{d}t.$$

In the last section, we saw that $\int (2t+4)\,\mathrm{d}t = t^2 + 4t + C$; in other words, $(2t+4)\,\mathrm{d}t = \mathrm{d}(t^2+4t)$. Therefore,

$$\begin{aligned}\int_3^5 (2t+4)\,\mathrm{d}t &= \int_3^5 \mathrm{d}(t^2+4t) = (t^2+4t)|_3^5 \\ &= \big((5)^2 + 4(5)\big) - \big((3)^2 + 4(3)\big) = (45) - (21) = 24.\end{aligned}$$

(Notice that this is the same answer as when I did this using geometry!)

This also explains why the same term 'integral' and symbol '$\int$' are used for both the definite integral (a sum of infinitely small quantities) and the indefinite integral (the antidifferential). They at first appear to be completely different concepts, but in reality they are closely related, through the Fundamental Theorem of Calculus.

4 *Integration techniques*

This leaves us with one problem: how do we find indefinite integrals?

Each rule for differentiation gives us a rule for integration. In the table below, I have some rules for differentiation (all of which you should know by now), together with corresponding rules for integration:

$$\begin{aligned}
\mathrm{d}(u+v) &= \mathrm{d}u + \mathrm{d}v, & \int (y+z)\,\mathrm{d}x &= \int y\,\mathrm{d}x + \int z\,\mathrm{d}x; \\
\mathrm{d}(ku) &= k\,\mathrm{d}u \text{ (when } k \text{ is constant)}, & \int ky\,\mathrm{d}x &= k\int y\,\mathrm{d}x \text{ (when } k \text{ is constant)}; \\
\mathrm{d}(uv) &= v\,\mathrm{d}u + u\,\mathrm{d}v, & \int u\,\mathrm{d}v &= uv - \int v\,\mathrm{d}u; \\
\mathrm{d}(u^n) &= nu^{n-1}\,\mathrm{d}u \text{ (when } n \text{ is constant)}, & \int u^m\,\mathrm{d}u &= \frac{1}{m+1}u^{m+1} + C \text{ (when } m \neq -1 \text{ is constant)}; \\
\mathrm{d}(\mathrm{e}^u) &= \mathrm{e}^u\,\mathrm{d}u, & \int \mathrm{e}^u\,\mathrm{d}u &= \mathrm{e}^u + C; \\
\mathrm{d}(\ln|u|) &= \frac{1}{u}\,\mathrm{d}u, & \int \frac{1}{u}\,\mathrm{d}u &= \ln|u| + C; \\
\mathrm{d}(\sin u) &= \cos u\,\mathrm{d}u, & \int \cos u\,\mathrm{d}u &= \sin u + C; \\
\mathrm{d}(\cos u) &= -\sin u\,\mathrm{d}u, & \int \sin u\,\mathrm{d}u &= -\cos u + C; \\
& \text{etc.}
\end{aligned}$$

Using these rules, you can work out all of the integrals in the textbook through Chapter 6, and then some.

For example, to find $\int (2t+4)\,\mathrm{d}t$:

$$\int (2t+4)\,\mathrm{d}t = \int 2t\,\mathrm{d}t + \int 4\,\mathrm{d}t = 2\int t^1\,\mathrm{d}t + 4\int \mathrm{d}t = 2\left(\frac{1}{2}t^2\right) + 4t + C = t^2 + 4t + C.$$

This is the same answer as we got before, but this time I didn't have to guess the answer and get lucky; I was able to actually calculate it. That's how you're going to be doing most of the problems.

For more complicated integrals, there are fancier techniques. Rather than learn all of these, you can program them into a computer. There are even free websites that will do this for you!

5 *Summary*

To find the indefinite integral $\int y\,\mathrm{d}x$, you need to use integration techniques; your answer will still have the variable in it and should end with a new local-constant term C. To find the definite integral $\int_a^b y\,\mathrm{d}x$, first find the indefinite integral and then take a difference; assuming a and b are constants, your answer will also be constant (and the C will disappear).

So for example, to find the definite integral of $2t + 4$ with respect to t from 3 to 5:

$$\int_3^5 (2t+4)\,\mathrm{d}t = \int_3^5 (2t^1\,\mathrm{d}t + 4\,\mathrm{d}t) = \left(2\left(\frac{1}{2}t^2\right) + 4t\right)\Big|_3^5 = (t^2+4t)|_3^5 = 45 - 21 = 24.$$

This is simply a combination of calculations that I did earlier, to find the indefinite integral and to apply the FTC.

6 *Semidefinite integrals*

Besides the *definite* integral $\int_a^b f(x)\,\mathrm{d}x$ and the *indefinite* integral $\int f(x)\,\mathrm{d}x$, there is also a **semidefinite integral** $\int_a f(x)\,\mathrm{d}x$. While the definite integral works out to a specific value (as long as f, a, and b are specified), the indefinite and semidefinite integrals still have the variable x in them. On the other hand, while the indefinite integral depends on an arbitrary C, the definite and semidefinite integrals don't have this. So the semidefinite integral fits in between the other two kinds.

Here is one way to define it:

$$\int_{x=a} f(x)\,\mathrm{d}x = \int_{t=a}^{x} f(t)\,\mathrm{d}t.$$

That is, introduce a new variable t and use the old variable x as the upper bound of a definite integal. The Second Fundamental Theorem of Calculus,

$$\int_{x=a}^{b} f(x)\,\mathrm{d}x = \left(\int f(x)\,\mathrm{d}x\right)\Big|_{x=a}^{b} = \left(\int f(x)\,\mathrm{d}x\right)\Big|_{x=b} - \left(\int f(x)\,\mathrm{d}x\right)\Big|_{x=a},$$

also tells us how to evaluate semidefinite integrals:

$$\int_{x=a} f(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x - \left(\int f(x)\,\mathrm{d}x\right)\Big|_{x=a}.$$

In other words, work out the indefinite integral as usual; then, instead of evaluating this at two values of the variable before subtracting, evalute it at one value and keep the variable in the other expression (then subtract). For example,

$$\int_{x=1} x\,\mathrm{d}x = \frac{x^2}{2} - \left(\frac{x^2}{2}\right)\Big|_{x=1} = \frac{x^2}{2} - \left(\frac{(1)^2}{2}\right) = \frac{1}{2}x^2 - \frac{1}{2}.$$

(You can probably skip the step with $|_{x=1}$ in it, since once you've written down $x^2/2$ before the minus sign, you can immediately plug in 1 for x to get $(1)^2\big/2$ after the minus sign.)

7 *Integration by parts*

Integration by parts is based on the Product Rule for differentiation. In terms of differentials, the Product Rule says that $\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v$. Taking indefinite integrals of both sides and rearranging the terms slightly, this becomes

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Unlike integration by substitution, you don't rewrite the problem in terms of u (nor v). Instead, you identify suitable u and v and their differentials and then write out the equation above in terms of x (or whatever your variable is).

You want to pick u and v so that $\int u\,\mathrm{d}v$ is the integral that you care about, which means splitting up the factors of the integrand, some into u and some into $\mathrm{d}v$. Once you know u and $\mathrm{d}v$, you can find $\mathrm{d}u$ and v, at least if you know how to integrate whatever $\mathrm{d}v$ is. (When you do this integration of $\mathrm{d}v$ to get v, you have a choice up to a local constant; you're deciding what v is, so just pick the simplest expression.) If you split things up well, then $\int v\,\mathrm{d}u$ will be simpler than what you started with.

Here is my advice on how to split factors into u and $\mathrm{d}v$ so that integration by parts will make the next integral easier. The items on the top of the list are the best choices for $\mathrm{d}v$, and the items on the bottom are the best choices for u. Put as many factors as you can into $\mathrm{d}v$, starting at the top of this list and working your way to the bottom, as long as you still have something that you know how to integrate to get v. Then put whatever factors are left over into u.

- $\mathrm{d}x$ (this *must* go into $\mathrm{d}v$),
- e^x and other exponential expressions,
- $\sin x$ and other trigonometric expressions,
- polynomials and other algebraic expressions,
- $\ln x$ and other logarithmic expressions,
- $\operatorname{asin} x = \sin^{-1} x$ and other inverse trigonometric expressions.

In complicated cases, you may have to use integration by parts more than once. Just keep going until either you get something that you can handle or you get back to where you started. In the latter case, you can set up an equation to solve for your integral.