Linear approximation

Given any function f and a number c in the domain of f, the **difference quotient** of f at c is a function $\tilde{f}_c$, given by

$$\tilde{f}_c(h) = \frac{f(c+h) - f(c)}{h}.$$

This is the function whose limit at 0 is the definition of the derivative. Note that $\tilde{f}_c$ is not defined at 0; however, if f is differentiable at c, then we can define $\tilde{f}_c(0)$ to be $f'(c)$. Then

$$f(c+h) = f(c) + \tilde{f}_c(h)\, h$$

for a function $\tilde{f}_c$ that's continuous at 0. Since $\tilde{f}_c$ is continuous at 0, we can say that $\tilde{f}_c(h) \approx \tilde{f}_c(0)$ when $h \approx 0$, or in other words, $\tilde{f}_c(h) \approx f'(c)$ when $h \approx 0$. Putting this approximation in the equation above, we get

$$f(c+h) \approx f(c) + f'(c)\, h$$

when $h \approx 0$. Writing x for $c+h$ (so that $h = x - c$), you can also put this as

$$f(x) \approx f(c) + f'(c)\,(x-c)$$

when $x \approx c$. While the left-hand side could be any differentiable function, the right-hand side is a linear function of x; this function is the **linear approximation** to f near c, or the **linearization** of f near c.

The textbook likes to name this function L; so $f(x) \approx L(x) = f(c) + f'(c)\,(x-c)$. I don't like that name, because which function you get as the linear approximation depends on which function you start with as well as on which number c you look at. So I write $L_{f,c}$ for the linearization of f near c:

$$f(x) \approx L_{f,c}(x) = f(c) + f'(c)\,(x-c).$$

This is actually only the beginning of a whole sequence of approximations, each (typically) better than the one before it:

$f(x) \approx f(c)$, a constant, if f is continuous at c;

$f(x) \approx f(c) + f'(c)\,(x-c)$, a linear function of x, if f is differentiable at c;

$f(x) \approx f(c) + f'(c)\,(x-c) + \frac{1}{2}\, f''(c)\,(x-c)^2$, a quadratic function of x, if f is twice differentiable at c;

$f(x) \approx f(c) + f'(c)\,(x-c) + \frac{1}{2}\, f''(c)\,(x-c)^2 + \frac{1}{6}\, f'''(c)\,(x-c)^3$, a cubic function of x, if f is 3-times differentiable at c;

$\vdots$

(This sequence of approximations is covered in Calculus 2; see Section 9.8 of the textbook and page 47 or so of the full set of notes.)

It's handy to describe linear approximation in terms of differentials and differences. While a differential represents an infinitesimal (infinitely small) change, a **difference** represents an appreciable or finitesimal (meaning *not* infinitely small) change. As x changes from c to $c+h$, we say that the difference in x is

$$\Delta x = (c+h) - c = h.$$

Meanwhile, if $y = f(x)$, then the difference in y is

$$\Delta y = y|_{x=c+h} - y|_{x=c} = f(c+h) - f(c).$$

To be specific, we can write

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} = f(c+h) - f(c).$$

Then the linear approximation says that

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} = f(c+h) - f(c) \approx f(c) + f'(c)\,h - f(c) = f'(c)\,h = \mathrm{d}y|_{\substack{x=c,\\ \mathrm{d}x=h}}.$$

So in the end, the linear approximation replaces differences with differentials. Although

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} \approx \mathrm{d}y|_{\substack{x=c,\\ \mathrm{d}x=h}}$$

is the proper way to put it, often one abbreviates this as

$$\Delta y \approx \mathrm{d}y.$$

(But really this only correct if we also have $\Delta x = \mathrm{d}x$, or at least $\Delta x \approx \mathrm{d}x$, because that difference is also replaced by a differential in the approximation.)

More generally, you can say that an equation involving differentials can be replaced by an approximate equation involving differences. For example, if $x^5 + 2x = y^5 + y$, then $5x^4\,\mathrm{d}x + 2\,\mathrm{d}x = 5y^4\,\mathrm{d}y + \mathrm{d}y$ (by differentiating both sides), so $5x^4\,\Delta x + 2\,\Delta x \approx 5y^4\,\Delta y + \Delta y$. Then if you are looking near the only obvious solution, $(x, y) = (0, 0)$, and you want to know the value of y when $x = 0.3$ (so $\Delta x = 0.3 - 0 = 0.3$, you find $5(0)^4(0.3) + 2(0.3) \approx 5(0)^4\,\Delta y + \Delta y$, so $\Delta y \approx 0.6$; in other words, the new y-value is approximately $0 + 0.6 = 0.6$. (The actual solution to $(0.3)^5 + 2(0.3) = y^5 + y$ is $y|_{x=0.3} \approx 0.55$ to 2 decimal places, but I couldn't do that by hand!)

It can be important to know how far off an approximation might be, and this is basically given by the next term in the sequence of approximations on the previous page. To be specific, the Mean-Value Theorem (see the next reading) says that $f(x) - f(c)$ (which is the error in the constant approximation $f(x) \approx f(c)$) cannot be any larger in absolute value than $|x - c|$ times the maximum value that f' takes between x and c; similarly, $f(x) - L_{f,c}(x)$ (which is the error in the linear approximation near c) cannot be any larger in absolute value than $|x - c|^2$ times half the maximum value that f'' takes between x and c. However, the details of why this is so are best saved for the full treatment of the entire sequence of approximations that begins on about page 47 of the full set of notes.