All of the integrals in vector calculus can be thought of as integrals of differential forms of one sort or another. Since integration of differential forms generalises in ways that integration of vector fields cannot (some of which are important in applications, especially to physics), it's useful to be able to think about differential forms. Furthermore, one then needs fewer formulas for all of the integrals.

General principles

Here I spell out the general principles of integrating differential forms, but it's really the examples that follow that will make the ideas clear.

There are three sorts of differential forms that we'll need: *oriented* forms, *pseudo-oriented* forms, and *unoriented* forms. The oriented forms are the most straightforward kind and the simplest to calculate with. The pseudo-oriented forms are essentially the same as oriented forms, except that their sign is determined by using the right-hand rule; if we used the left-hand rule instead, then the pseudo-oriented forms would have opposite sign but the results of all integrals would stay the same. (It is sometimes handy to keep track of whether something is oriented or pseudo-oriented, but you can ignore the difference as long as you always use the right-hand rule.) The unoriented forms are least used in applications; they typically arise by taking the absolute value of another form (and then possibly multiplying by a scalar).

We'll integrate these forms along various regions in space, called *manifolds*. These manifolds can also be oriented, pseudo-oriented, or unoriented. Now the unoriented manifolds are the simplest; they are just shapes. With an oriented manifold, we also make a choice of what direction to go along the manifold; with a pseudo-oriented manifold, we instead make a choice of what direction to go around or across the manifold. As you might guess, we integrate oriented forms on oriented manifolds, pseudo-oriented forms on pseudo-oriented manifolds, and unoriented forms on unoriented manifolds.

Our manifolds will be *parametrised*; we'll have one or more variables $t, u, v, \ldots$ (the *parameters*), taking a finite range of values, and a function (the *parametrisation*) specifying which point in space corresponds to which values of the parameters. Running this function over the entire range of the parameters carves out the manifold. (We'll want our parametrisation functions to be continuously differentiable, in order to avoid technicalities about whether the integrals are defined. Similarly, the forms themselves should be continuous.)

The number of parameters used is the *dimension* of the manifold. This must match the *rank* of the differential form, which is the number of differentials in each term of the form. These differentials are combined using the *wedge product*, $\wedge$. A key property of the wedge product is that it is *anticommutative* between differentials; that is,

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$$

(much like the cross product of vectors). This also means that $\mathrm{d}x \wedge \mathrm{d}x = 0$. However, for unoriented forms, we take the absolute value of the wedge product; then $|\mathrm{d}x \wedge \mathrm{d}y| = |-\mathrm{d}y \wedge \mathrm{d}x| = |\mathrm{d}y \wedge \mathrm{d}x|$, while $|\mathrm{d}x \wedge \mathrm{d}x| = |0| = 0$ still.

To calculate the integral, you use the parametrisation to express the coordinates $x, y, \ldots$ in terms of the parameters $t, u, v, \ldots$, and differentiate this to get $\mathrm{d}x, \mathrm{d}y, \ldots$ in terms of $\mathrm{d}t, \mathrm{d}u, \mathrm{d}v, \ldots$, so that the integral is entirely in terms of the parameters. We then express this as an iterated integral, making sure that the (pseudo)-orientation (if any) matches (or putting a minus sign out front if it doesn't).

Curves

A **curve** C is a manifold of dimension 1. So it is given by a vector-valued function $\vec{r} = \langle x, y, \ldots \rangle$ of one variable t (which we'll assume is continuously differentiable). When we orient a curve, we specify which direction to travel along the curve; when we pseudo-orient a curve in 2 dimensions, we specify which direction to travel across the curve. (We won't need to pseudo-orient a curve in more dimensions.)

To integrate a vector field $\vec{F} = \langle M, N, \ldots \rangle$ along an oriented curve C, we integrate the rank-1 oriented form $\vec{F} \cdot \mathrm{d}\vec{r}$:

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \langle M, N, \ldots \rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \ldots \rangle = \int_C (M\,\mathrm{d}x + N\,\mathrm{d}y + \cdots) = \int_C \left(M \frac{\mathrm{d}x}{\mathrm{d}t} + N \frac{\mathrm{d}y}{\mathrm{d}t} + \cdots \right) \mathrm{d}t$$

or

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_C \langle M, N, \ldots\rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots \right\rangle \mathrm{d}t = \int_C \left(M\frac{\mathrm{d}x}{\mathrm{d}t} + N\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots \right) \mathrm{d}t.$$

(There's no need to learn all of these formulas; just learn one, then put everything in terms of t and push through.) To match orientations, make sure that the diretion along the curve as t increases is the same direction as the curve's orientation; otherwise put a minus sign out front.

To integrate a vector field $\vec{F} = \langle M, N\rangle$ across a pseudo-oriented curve C in 2 dimensions, we integrate the rank-1 pseudo-oriented form $\vec{F} \times \mathrm{d}\vec{r}$ (where the cross product in 2 dimensions produces a scalar, or rather a pseudo-scalar since the sign depends on the right-hand rule):

$$\int_C \vec{F} \times \mathrm{d}\vec{r} = \int_C \langle M, N\rangle \times \langle \mathrm{d}x, \mathrm{d}y\rangle = \int_C (M\,\mathrm{d}y - N\,\mathrm{d}x) = \int_C \left(M\frac{\mathrm{d}y}{\mathrm{d}t} - N\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t$$

or

$$\int_C \vec{F} \times \mathrm{d}\vec{r} = \int_C \vec{F} \times \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_C \langle M, N\rangle \times \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t} \right\rangle \mathrm{d}t = \int_C \left(M\frac{\mathrm{d}y}{\mathrm{d}t} - N\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t.$$

To match pseudo-orientations, make sure that the direction along the curve as t increases is counterclockwise from the direction of the curve's pseudo-orientation; otherwise put a minus sign out front.

To integrate a scalar field f on an unoriented curve C, we integrate the rank-1 unoriented form $f\,\mathrm{d}s$, where s has no meaning by itself but instead $\mathrm{d}s$ is the unoriented form $\|\mathrm{d}r\|$:

$$\int_C f\,\mathrm{d}s = \int_C f\,\|\mathrm{d}\vec{r}\| = \int_C f\,\|\langle \mathrm{d}x, \mathrm{d}y, \ldots\rangle\| = \int_C f\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + \cdots} = \int_C f\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \cdots}\,|\mathrm{d}t|$$

or

$$\int_C f\,\mathrm{d}s = \int_C f\,\|\mathrm{d}\vec{r}\| = \int_C f\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\|\,|\mathrm{d}t| = \int_C f\left\|\left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots \right\rangle\right\|\,|\mathrm{d}t| = \int_C f\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \cdots}\,|\mathrm{d}t|.$$

Now there is no orientation to match; instead, make sure that t increases, so that $|\mathrm{d}t| = \mathrm{d}t$.

Surfaces

A **surface** R is a manifold of dimension 2. So it is given by a vector-valued function $\vec{r} = \langle x, y, \ldots\rangle$ of two variables u, v (which we'll assume is continuously differentiable). When we pseudo-orient a surface in 3 dimensions, we specify which direction to travel across the curve. (We won't need to pseudo-orient a surface in more dimensions, nor will we orient any at all.)

To integrate a vector field $\vec{F} = \langle M, N, O\rangle$ across a pseudo-oriented surface R in 3 dimensions, we integrate the rank-2 pseudo-oriented form $\vec{F} \cdot \mathrm{d}\vec{S}$, where $\vec{S}$ has no meaning by itself, but instead $\mathrm{d}\vec{S}$ is the psuedo-vector-valued form $1/2\,\mathrm{d}\vec{r} \mathbin{\hat{\wedge}} \mathrm{d}\vec{r}$ (which as a vector is multiplied by the cross product and as a differential form is multiplied by the wedge product), which works out to $\langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y\rangle$ (using the right-hand rule) or $\partial\vec{r}/\partial u \times \partial\vec{r}/\partial v\,\mathrm{d}u \wedge \mathrm{d}v$:

$$\begin{aligned}\int_R \vec{F} \cdot \mathrm{d}\vec{S} &= \int_R \langle M, N, O\rangle \cdot \langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y\rangle = \int_R (M\,\mathrm{d}y \wedge \mathrm{d}z + N\,\mathrm{d}z \wedge \mathrm{d}x + O\,\mathrm{d}x \wedge \mathrm{d}y)\\ &= \int_R \left(M\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + N\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + O\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) \right) \mathrm{d}u \wedge \mathrm{d}v\end{aligned}$$

or

$$\begin{aligned}\int_R \vec{F} \cdot \mathrm{d}\vec{S} &= \int_R \langle M, N, O\rangle \cdot \frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\,\mathrm{d}u \wedge \mathrm{d}v = \int_R \langle M, N, O\rangle \cdot \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \mathrm{d}u \wedge \mathrm{d}v\\ &= \int_R \left(M\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + N\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + O\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) \right) \mathrm{d}u \wedge \mathrm{d}v.\end{aligned}$$

To match pseudo-orientations, make sure that, as you curl from the direction in which u increases towards the direction in which v increases, the right-hand rule gives the direction of the surface's pseudo-orientation; otherwise put a minus sign out front.

To integrate a scalar field f on an unoriented surface R, we integrate the rank-2 unoriented form $f\,\mathrm{d}\sigma$, where σ has no meaning by itself but instead $\mathrm{d}\sigma$ is the unoriented form $\|\mathrm{d}S\|$:

$$\begin{aligned}\int_R f\,\mathrm{d}\sigma &= \int_R f\,\|\mathrm{d}\vec{S}\| = \int_R f\,\|\langle \mathrm{d}y\wedge\mathrm{d}z, \mathrm{d}z\wedge\mathrm{d}x, \mathrm{d}x\wedge\mathrm{d}y\rangle\| = \int_R f\sqrt{(\mathrm{d}y\wedge\mathrm{d}z)^2+(\mathrm{d}z\wedge\mathrm{d}x)^2+(\mathrm{d}x\wedge\mathrm{d}y)^2}\\ &= \int_R f\sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}\,|\mathrm{d}u\wedge\mathrm{d}v|\end{aligned}$$

or

$$\begin{aligned}\int_R f\,\mathrm{d}\sigma &= \int_R f\,\|\mathrm{d}\vec{S}\| = \int_R f\left\|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right\||\mathrm{d}u\wedge\mathrm{d}v| = \int_R f\left\|\left\langle\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u}\right\rangle\times\left\langle\frac{\partial x}{\partial v},\frac{\partial y}{\partial v},\frac{\partial z}{\partial v}\right\rangle\right\||\mathrm{d}u\wedge\mathrm{d}v|\\ &= \int_R f\sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}\,|\mathrm{d}u\wedge\mathrm{d}v|.\end{aligned}$$

Again there is no orientation to match; instead, make sure that u and v both increase, so that $|\mathrm{d}u\wedge\mathrm{d}v| = \mathrm{d}u\,\mathrm{d}v$.

The Stokes Theorem

The (second) Fundamental Theorem of Calculus states that

$$\int_a^b \mathrm{d}f = f|_a^b.$$

This works just as well when there are several independent variables as when there is just one. In this case, we can also write $\mathrm{d}f$ as $\nabla f\cdot\mathrm{d}\vec{r}$ to get the theorem

$$\int_a^b \nabla f\cdot\mathrm{d}\vec{r} = f|_a^b.$$

Although this is now a theorem about integrating a gradient along a curve, in essence it is still just the FTC, a theorem about integrating differentials.

This theorem generalises to differential forms of higher rank, where it is called the **Stokes Theorem**. To do this properly, we need to know two things: how to take the differential of a differential form, and how to take the endpoints of manifold other than a curve.

With endpoints, we're really dealing with the *boundary* of a manifold. The boundary of a curve oriented from a to b consists of both $\{a\}$ and $\{b\}$, the former negatively and the latter positively. If you think of a point $\{a\}$ as a manifold of dimension 0 and think of a scalar quantity f as a differential form of rank 0, then we integrate f on $\{a\}$ by simply taking the value of f at a: $\int_{\{a\}} f = f|_a$. Then the FTC can be written as

$$\int_C \mathrm{d}f = \int_{\partial C} f,$$

where the symbol '∂' indicates the boundary. Then the boundary of a surface is a curve, and the boundary a bounded region of space is a surface.

When we take the differential of a differential form, we get another differential form if we take the *exterior* differential; the exterior differential of a form ω is $\mathrm{d}\wedge\omega$. When we add forms, the exterior differential obeys the Sum Rule as usual; when we multiply them, we have a kind of Product Rule too. This is the same as the usual Product Rule, except that we must keep track of the order of multiplication, and also remember to insert a minus sign when we reverse the order of two differentials. Finally, the exterior differential of a differential is zero. For example,

$$\mathrm{d}\wedge(x\,\mathrm{d}y) = \mathrm{d}x\wedge\mathrm{d}y + x\,\mathrm{d}\wedge\mathrm{d}y = \mathrm{d}x\wedge\mathrm{d}y + 0 = \mathrm{d}x\wedge\mathrm{d}y.$$

When we relate differential forms to vector fields, we'll also use various ways of taking derivatives of vector fields. These can be expressed using ∇ and one of the ways of multiplying vectors: the **divergence** $\nabla \cdot \vec{F}$ is a scalar field, and the **curl** $\nabla \times \vec{F}$ is a pseudo-vector field in 3 dimensions or a pseudo-scalar field in 2 dimensions. Specifically,

$$\nabla \cdot F = \langle \partial/\partial x, \partial/\partial y, \ldots \rangle \cdot \langle M, N, \ldots \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \cdots,$$

and

$$\nabla \times \vec{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \times \langle M, N, O \rangle = \left\langle \frac{\partial O}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial O}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

in 3 dimensions, or

$$\nabla \times \vec{F} = \langle \partial/\partial x, \partial/\partial y \rangle \times \langle M, N \rangle = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

in 2 dimensions.

Now suppose that a surface R is bounded by a curve $\partial R = C$. The Stokes Theorem tells us that

$$\int_R \mathrm{d}\alpha = \int_{\partial R} \alpha,$$

where α is a differential form of rank 1. If I integrate a vector field $\vec{F}$ along C, then I'm really integrating the differential form $\vec{F} \cdot \mathrm{d}\vec{r}$, so

$$\int_{\partial R} \vec{F} \cdot \mathrm{d}\vec{r} = \int_R \mathrm{d}\big(\vec{F} \cdot \mathrm{d}\vec{r}\big).$$

But in 3 dimensions, this last quantity is the same as

$$\int_{\partial R} \vec{F} \cdot \mathrm{d}\vec{r} = \int_R \nabla \times \vec{F} \cdot \mathrm{d}S;$$

and in 2 dimensions, it's the same as

$$\int_{\partial R} \vec{F} \cdot \mathrm{d}\vec{r} = \int_R \nabla \times \vec{F} \, \mathrm{d}A,$$

where $\mathrm{d}A$ is the area form $\mathrm{d}x \wedge \mathrm{d}y$. These are the theorems traditionaly called *Stokes's Theorem* and *Green's Theorem*, respectively. If, in 2 dimensions, I integrate $\vec{F}$ across C, then

$$\int_{\partial R} \vec{F} \times \mathrm{d}\vec{r} = \int_R \nabla \cdot \vec{F} \, \mathrm{d}A$$

is another form of Green's Theorem.

Next, suppose that a bounded region Q in space is bounded by a surface $\partial Q = R$. Now the Stokes Theorem tells us that

$$\int_Q \mathrm{d}\alpha = \int_{\partial Q} \alpha,$$

where now α is a differential form of rank 2. If I integrate a vector field $\vec{F}$ across R, then I'm realling integrating $\vec{F} \cdot \mathrm{d}S$, so

$$\int_{\partial Q} \vec{F} \cdot \mathrm{d}S = \int_Q \mathrm{d}\big(\vec{F} \cdot \mathrm{d}S\big).$$

But in 3 dimensions, this last quantity is the same as

$$\int_{\partial Q} \vec{F} \cdot \mathrm{d}S = \int_Q \nabla \cdot \vec{F} \, \mathrm{d}V,$$

where $\mathrm{d}V$ is the volume form $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. This is the theorem traditionally called *Gauss's Theorem.*