Differential forms are, broadly speaking, expressions that may have *differentials* in them. They have many uses in modern science and engineering, even though they are not traditionally covered explicitly in math class. They are covered somewhat, however, and they are there whenever you differentiate or integrate, even if you don't recognize them. They are especially prominent in multivariable calculus, and I want to bring them to your attention; you'll find that symbols that otherwise seem meaningless or merely mnemonic can be understood literally (or almost literally) as differential forms.

Examples

The most basic examples of differential forms are differentials such as $\mathrm{d}x$ and $\mathrm{d}y$. In general, if u is any quantity that might change, then $\mathrm{d}u$ is intended to be a related quantity whose value is an infinitely small change in u, or rather the amount by which the value of u changes when an infinitely small (or arbitrarily small) change is made. (I will make this precise later on.)

Besides the differentials themselves, differential forms can be constructed by applying arithmetic operations, so $\mathrm{d}x + \mathrm{d}y$, $\mathrm{d}x\,\mathrm{d}y$, and $\sqrt{\mathrm{d}x}$ are all differential forms. In all of these expressions, we adopt an order of operations in which the differential operator d is applied before any arithmetic operator; for example, $\mathrm{d}x^2$ means $(\mathrm{d}x)^2$, not $\mathrm{d}(x^2)$ (which is $\mathrm{d}u$ when $u = x^2$). Additionally, we can include ordinary quantities in these expressions, so $x + \mathrm{d}x$, $3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$, and $x \ln{(y/\mathrm{d}z)}$ are also differential forms. We can also use differentials of differentials, such as d^2x (the differential of $\mathrm{d}x$), although we won't need such *higher differentials* in this course. Besides all of this, any ordinary expression counts as a differential form in a degenerate way; thus, x, y^2, and $2xy^3$ are also differential forms.

Some differential forms are more useful than others. Of those listed above, besides the differentials and the non-differential quantities, the ones most likely to appear in a real problem are $\mathrm{d}x + \mathrm{d}y$ and $3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$. These consist of any number of terms, each of which is the product of an ordinary quantity (possibly 1) and the differential of an ordinary quantity. Differential forms with this property are most commonly found in practice. However, we will use other differential forms, such as $3x\,|\mathrm{d}y|$ and $\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$; however, you might be able to see how even these forms are differential of *degree* 1 in a sense similar to the degree of a polynomial.

All of the examples so far are differential forms of *order* 1; there are also differential forms of higher order, such as $\mathrm{d}x \wedge \mathrm{d}y$, which are written using a new operation, the *wedge product*. We will not use these until later; these notes are only about differential forms of order 1, or 1-forms for short. (Ordinary quantities may be viewed as differential forms of any order but are most commonly thought of as 0-forms, that is differential forms of order 0. This is why they are useful despite not having degree 1. In general, the useful differential forms have the same degree as their order, and people who study differential forms most often study the so-called *exterior* differential forms, for which the degree and order automatically match. However, this leaves out many useful differential forms, such as $3x\,|\mathrm{d}y|$, some of which we will need in this course.)

Evaluating differential forms

In this class, we generally assume that any ordinary quantity (that is any 0-form) is a function of 2 or 3 ordinary variables, $R = (x, y)$ or $R = (x, y, z)$. Thus, we evaluate ordinary quantites (0-forms) by specifying specific values for the variables that comprise R. For example, to evaluate $u = x^2 + xy$ when $x = 2$ and $y = 3$, we may write

$$u|_{R=(2,3)} = (x^2 + xy)|_{(x,y)=(2,3)} = (2)^2 + (2)(3) = 10.$$

To evaluate a differential form, we need not only a point (a value of R) but also a vector (a value of $\mathrm{d}R$). So for example, to evaluate $\alpha = 3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$ when $x = 2$, $y = 3$, $z = 4$, $\mathrm{d}x = 0.05$, $\mathrm{d}y = -0.01$, and $\mathrm{d}z = 0$, we may write

$$\begin{aligned} \alpha|_{R=(2,3,4),\mathrm{d}R=\langle 0.05,-0.01,0\rangle} &= (3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z)|_{(x,y,z)=(2,3,4),\langle \mathrm{d}x,\mathrm{d}y,\mathrm{d}z\rangle=\langle 0.05,-0.01,0\rangle} \\ &= 3(0.05) + (2)^2(-0.01) + \mathrm{e}^3(0) = 0.11. \end{aligned}$$

(Differential forms are often denoted with Greek letters, but they don't have to be.) We say that α has been evaluated at the point $R = (2, 3, 4)$ along the vector $\mathrm{d}R = \langle 0.05, -0.01, 0\rangle$. (The components of $\mathrm{d}R$ don't need to be small, since the definition is purely formal, but in applications that's what matters; after all, $\mathrm{d}R$ is supposed to be a *small* change in position.)

For differential forms that involve higher differentials, we need an additional vector $\mathrm{d}^2R = \langle \mathrm{d}^2x, \mathrm{d}^2y\rangle$, etc. In general, we may need an entire curve through a given point, but we won't need that level of generality in this course.

Differential forms as vectors

A differential form $\alpha = M\,\mathrm{d}x + N\,\mathrm{d}y + O\,\mathrm{d}z$ may be viewed as a dot product $\alpha = \langle M, N, O\rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\rangle = \mathbf{V} \cdot \mathrm{d}R$. For example, if $\alpha = 3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$, then $\alpha = \langle 3, x^2, \mathrm{e}^y\rangle \cdot \mathrm{d}R$; conversely, if $\mathbf{V} = \langle 3, x^2, \mathrm{e}^y\rangle$, then

$$\mathbf{V} \cdot \mathrm{d}R = \langle 3, x^2, \mathrm{e}^y\rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\rangle = 3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z.$$

We can recover $\mathbf{V}$ from α formally by evaluating α when $\mathrm{d}R$ is $\langle \mathbf{i}, \mathbf{j}\rangle$ or $\langle \mathbf{i}, \mathbf{j}, \mathbf{k}\rangle$, but there's probably no need to think about that explicitly.

Even in circumstances where it makes no sense to interpret a change in the values of (x, y, z) as a vector in the geometric sense (with length and direction), in which case dot products involving them generally have no meaning, it is traditional to write differential forms in this way and to focus on $\mathbf{V}$ rather than on α as the object of study. In this case, one sometimes refers to $\mathbf{V}$ as a *row vector*, distinguishing it from the usual notion of vector as a change in position, which in this context is called a *column vector*. (This terminology comes from matrix theory, but that really doesn't matter for us.) Row vectors may meaningfully be added together and multiplied by scalars, but they should be multiplied only by column vectors and only using the dot product. In particular, adding a row vector to a point to get a new point makes no sense generally.

Regardless of whether $\mathbf{V}$ has geometric significance as a vector, it can be helpful to visualize it as one. When calculations with a row vector need to be performed, however, ultimately it is the differential form $\alpha = \mathbf{V} \cdot \mathrm{d}R$ that matters. It's more common to see $\mathbf{V} \cdot \mathrm{d}\mathbf{r}$; the vector $\mathbf{r} = R - O$ (where O is $(0, 0)$ or $(0, 0, 0)$, the origin of the coordinate system) satisfies $\mathrm{d}\mathbf{r} = \mathrm{d}R$ (since O is constant). Sometimes $\mathbf{V} \cdot \mathrm{d}\mathbf{r}$ is even regarded as merely a mnemonic notation, but it can be taken literally, just as $\mathrm{d}y/\mathrm{d}x$ (which is also sometimes regarded as merely mnemonic) can be viewed literally as the result of a division of differentials. In any case, people do write $\mathbf{V} \cdot \mathrm{d}\mathbf{r}$, so it can be nice to know what it means!

In 2 dimensions, we can also take cross products, using the rule $\langle a, b\rangle \times \langle c, d\rangle = ad - bc$. For example, if $\mathbf{V} = \langle x, y\rangle$, then

$$\mathbf{V} \times \mathrm{d}\mathbf{r} = \langle x, y\rangle \times \langle \mathrm{d}x, \mathrm{d}y\rangle = x\,\mathrm{d}y - y\,\mathrm{d}x.$$

However, this requires that changes in x and y make sense as having a geometric length even when $\mathbf{V}$ is regarded as merely a row vector, so it doesn't come up as often.