This exam ran from 10:00 to 11:25 in the morning; if I wrote it well, then you didn't need the whole time. You may use any notes that you've written yourself, as well as quizzes that I've graded and returned to you, but not your textbook or anything else not written by you, and you may not communicate with anybody but me. You can come up and talk to me if you have questions, especially about the instructions. Also, you may use a calculator if you wish, although you shouldn't need one.

Take your time, and check over your answers. Read the instructions carefully, and be sure to show everything that they ask. You can always show *more* work if you like; for instance, you may draw a graph if you find it helpful, even when a problem does not require you to draw a graph. If you're unsure of your answer, then explain what you're unsure about, and show your work so that you can get as much partial credit as possible.

Most of these problems consists of several related parts. You don't have to do those parts in order one after another; depending on how you think, it may be easier to do them in a different order or even to go back and forth. But make sure that the final answer to each part is clear.

Don't forget to put your name on the exam!!!

1 *Consider the equation* $y = \frac{x^2}{2}$.

a *Draw a graph of this equation; try to make it clear, and be sure to label the scale. To show your work, either show a table of values that you calculate or explain how your graph has been modified using a coordinate transformation. (If you know some other method to draw the graph, you may use it if you explain briefly what you're doing. But* don't *just copy the graph from a graphing calculator!)*

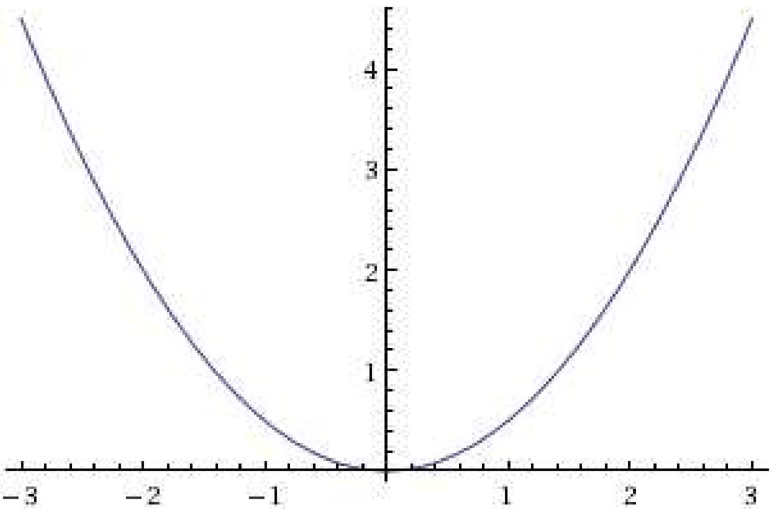
One way to do this is to pick some values for x and make a table. I'll pick $x = -2, -1, 0, 1, 2$, but you could have picked different arguments. Then my table of values is

x	$y = x^2/2$
-2	$(-2)^2/2 = 2$
-1	$(-1)^2/2 = 1/2$
0	$(0)^2/2 = 0$
1	$(1)^2/2 = 1/2$
2	$(2)^2/2 = 2$

Then I plot the points $(-2, 2)$, $(-1, 1/2)$, $(0, 0)$, $(1, 1/2)$, and $(2, 2)$ and connect them smoothly from left to right with arrows at the ends.

Another way to do this is to transform the graph of $y = x^2$. Since I divide by 2 after the squaring, the graph is compressed vertically by a factor of 2. So I start with the points $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 4)$, then divide each second coordinate by 2 to get $(-2, 2)$, $(-1, 1/2)$, $(0, 0)$, $(1, 1/2)$, and $(2, 2)$. I plot these points and connect them with a curve with the same basic shape as the graph of $y = x^2$.

In either case, here is my graph on the next page:



(This graph was produced by Wolfram Alpha. There really should be arrows at the ends of the curve.)

b *Is this graph symmetric with respect to the horizontal axis? with respect to the vertical axis? with respect to the origin? Answer each question Yes or No. (Either show how to calculate these algebraically or give answers that match your graph from part a.)*

Looking at the graph, it is **not symmetric** with respect to the horizontal axis, **symmetric** with respect to the vertical axis, and **not symmetric** with respect to the origin.

You could also calculate this using the equation without using the graph.

- Changing y to $-y$, $y = x^2/2$ becomes $-y = x^2/2$, which simplifies to $y = -x^2/2$. Since this is different from the original, the graph is **not symmetric** with respect to the horizontal axis.
- Changing x to $-x$, $y = x^2/2$ becomes $y = (-x)^2/2$, which simplifies to $y = x^2/2$. Since this is the same as the original, the graph is **symmetric** with respect to the vertical axis.
- Changing both, $y = x^2/2$ becomes $-y = (-x)^2/2$, which simplifies to $y = -x^2/2$. Since this is different from the original, the graph is **not symmetric** with respect to the origin.

c *Is the function* $f(x) = \dfrac{x^2}{2}$ *even, odd, or neither? (Either show how to calculate this algebraically or give an answer that matches your graph from part a.)*

The graph of this function is the same as the graph above; since this graph is symmetric with respect to the vertical axis (but not the origin), the function is **even**.

Again, you could calculate this using the formula:

$$f(-x) = \frac{(-x)^2}{2} = \frac{x^2}{2} = f(x),$$

so f is **even**.

2 *Consider the points* $(2, -1)$ *and* $(-4, 3)$.

The run from the first point to the other is $(-4) - (2) = -6$, and the rise is $(3) - (-1) = 4$. (I'll use these in the rest of this problem.)

a *Find the distance between these two points. (Show what numerical calculation you make or what equation you solve.)*

To find the distance, I square the run and the rise, add the results, and then take the square root:

$$\sqrt{(-6)^2 + (4)^2} = \sqrt{(36) + (16)} = \sqrt{52} = 2\sqrt{13}.$$

b *Find the slope of the line through these two points. (Show what numerical calculation you make or what equation you solve.)*

To find the slope, I divide the rise by the run:

$$\frac{4}{-6} = -\frac{2}{3}.$$

c *Find both intercepts of this line. (Show what numerical calculation you make or what equation you solve.)*

Since the slope exists, there is an equation of the form

$$y = mx + b,$$

where m is the slope. Using the result of part (b) and the coordinates of the first point, I get

$$-1 = -\frac{2}{3}(2) + b;$$
$$-1 = -\frac{4}{3} + b;$$
$$\frac{1}{3} = b.$$

Therefore, the equation is

$$y = -\frac{2}{3}x + \frac{1}{3}.$$

From this, I get one intercept immediately:

$$\left(0, \frac{1}{3}\right).$$

To get the other,

$$0 = -\frac{2}{3}x + \frac{1}{3};$$
$$\frac{2}{3}x = \frac{1}{3};$$
$$x = \frac{1}{2},$$

giving me

$$\left(\frac{1}{2}, 0\right).$$

3 *Consider these equations:*

$$4x - 2y = 4,$$
$$2x + 4y = 14.$$

a *Are the graphs of these equations parallel, perpendicular, or neither? (State what numbers you compare to determine this.)*

These are linear equations, so I'll solve each equation for y to find the slope of its graph. First,

$$\begin{aligned} 4x - 2y &= 4; \\ -2y &= -4x + 4; \\ y &= 2x - 2. \end{aligned}$$

Next,

$$\begin{aligned} 2x + 4y &= 14; \\ 4y &= -2x + 14; \\ y &= -\frac{1}{2}x + \frac{7}{2}. \end{aligned}$$

Therefore, the slopes are 2 and $-1/2$. Since these are different, the lines are *not* parallel; however, since their product is -1, the lines are **perpendicular** instead.

b *Solve the system of equations that consists of these two equations. (Show enough work that I can tell which method you are using.)*

Since I already solved these equations for y, I may as well use substitution. The first equation gave me a simpler result:

$$y = 2x - 2.$$

I substitute this into the second equation (in its original form):

$$\begin{aligned} 2x + 4y &= 14; \\ 2x + 4(2x - 2) &= 14; \\ 2x + 8x - 8 &= 14; \\ 10x &= 22; \\ x &= \frac{11}{5}. \end{aligned}$$

Then

$$y = 2x - 2 = 2\left(\frac{11}{5}\right) - 2 = \frac{22}{5} - 2 = \frac{12}{5}.$$

In other words,

$$(x, y) = \left(\frac{11}{5}, \frac{12}{5}\right).$$

4 *Given each of the following functions f, find $f(0)$. (Show at least one intermediate step for each; for part c, this may simply be marking the relevant portion of the graph. Note that these three parts are completely independent of each other.)*

a $f(x) = x^2 - 4$

Changing x to 0 in the formula for f, I find that

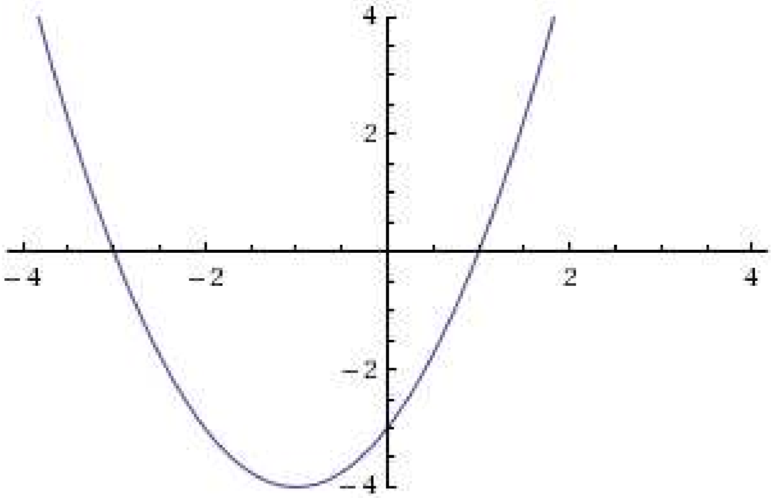
$$f(0) = (0)^2 - 4 = 0 - 4 = -4.$$

b $f(x) = \begin{cases} 5 - x & \text{for } x \leq 0, \\ 7 & \text{for } 0 < x < 1, \\ 4x + 5 & \text{for } x \geq 1 \end{cases}$

Since $0 \leq 0$, I find that

$$f(0) = 5 - (0) = 5.$$

c *f is the function with this graph:*



Since $(0, -3)$ is a point on this graph, I find that

$$f(0) = -3.$$

5 *Consider the functions f and g defined by these formulas:*

$$f(x) = 5x + 4,$$
$$g(x) = 4x + 5.$$

Write down and simplify formulas for the following functions. (Show at least one intermediate step for each.)

a $f - g$

I subtract the formula for g from the formulas for f:

$$(f - g)(x) = f(x) + g(x) = (5x + 4) - (4x + 5) = 5x + 4 - 4x - 5 = x - 1.$$

b $f \circ g$

I put the formula for g inside the formula for f:

$$(f \circ g)(x) = f\big(g(x)\big) = f(4x + 5) = 5(4x + 5) + 4 = 20x + 25 + 4 = 20x + 29.$$

c f^{-1}

I'll start with $f(x) = y$ and solve for x to find $f^{-1}(y)$:

$$\begin{aligned} 5x + 4 &= y; \\ 5x &= y - 4; \\ x &= \frac{1}{5}y - \frac{4}{5}. \end{aligned}$$

Therefore,

$$f^{-1}(y) = \frac{1}{5}y - \frac{4}{5}.$$

6 *Consider the function g defined by*

$$g(x) = \sqrt{x+1}.$$

a *What is the domain of g? (Either show what equation or inequality you solve to find this or draw a graph which shows your answer.)*

Addition is always defined, but I can't take a square root of a negative number. Thus,

$$\begin{aligned} x + 1 &\geq 0; \\ x &\geq -1. \end{aligned}$$

Therefore,

$$\operatorname{dom} g = \{x \mid x \geq -1\}.$$

b *What is the range of g? (Either show what equation or inequality you solve to find this or draw a graph which shows your answer.)*

The range of the square-root function is $[0, \infty)$, and g is transformation of this that is shifted to the left by 1. Since this affects only the domain, not the range,

$$\operatorname{ran} g = [0, \infty)$$

too.

c *What is the average rate of change of g from 2 to 5? (Show what numerical calculation you make.)*

First,

$$\begin{aligned} g(2) &= \sqrt{(2) + 1} = \sqrt{3}; \\ g(5) &= \sqrt{(5) + 1} = \sqrt{6}. \end{aligned}$$

Therefore, the average rate of change is

$$\frac{(\sqrt{6}) - (\sqrt{3})}{(5) - (2)} = \frac{\sqrt{6} - \sqrt{3}}{3}.$$

7 Extra credit.

Using a central-pivot irrigation system, a farmer irrigates a circular patch within a square field. The size of the field is fixed; the farmer irrigates the largest possible circle within that field. Express the area of ground that the system irrigates as a function of the area of the field. (Show what equations you solve to find this.)

Hints: The area of a cirlce is πr^2, where π is a constant and r is the radius of the circle (the distance from its centre to its edge). The area of a square is l^2, where l is the length of any side of the square.

(You may keep the constant π in your final answer, or you may use the approximation $\pi \approx 3.14$ if you wish.)

Let x be the area of the square field, and let y be the area of the circular patch. Then

$$x = l^2,$$
$$y = \pi r^2,$$
$$l = 2r.$$

Since I want y as a function of x, I solve these equations for the variables which are *not* x:

$$l = \sqrt{x},$$
$$r = \frac{l}{2} = \frac{\sqrt{x}}{2},$$
$$y = \pi \left(\frac{\sqrt{x}}{2} \right)^2 = \frac{\pi x}{4}.$$

Therefore, the area of the circular patch is $\pi/4$ times the area of the square field. (This is about 78.5 %.)