This exam runs from 12:30 to 2:00; if I've written it well, then you shouldn't need the whole time. You may use one sheet of notes that you've written yourself, but not your textbook or anything else not written by you, and you may not communicate with anybody but me. You can come up and talk to me if you have questions, especially about the instructions. Also, you may use a calculator if you wish, although you shouldn't need one.

Take your time, and check over your answers. Read the instructions carefully, and be sure to show everything that they ask. You can always show *more* work if you like; for instance, you may draw a graph if you find it helpful, even when a problem does not require you to draw a graph. If you're unsure of your answer, then explain what you're unsure about, and show your work so that you can get as much partial credit as possible.

Don't forget to put your name on the exam!!!

1 *Consider a circle in the (x, y)-plane whose diameter runs from $(7, 2)$ to $(3, 5)$.*

a *What is the centre of this circle? (Show what numerical calculation you make or what equation you solve.)*

The centre is halfway between these points, so I take the average:

$$\left(\frac{7+3}{2}, \frac{2+5}{2}\right) = \left(5, \frac{7}{2}\right).$$

b *What is the radius of this circle? (Show what numerical calculation you make or what equation you solve.)*

First, the diameter is the distance between these points, which I find using the Pythagorean Theorem:

$$\sqrt{(7-3)^2 + (2-5)^2} = 5.$$

Then, the radius is half the diameter:

$$\frac{5}{2}.$$

c *Write down an equation for this circle.*

Since the centre is $(5, 7/2)$ and the radius is $5/2$, an equation for the circle is

$$(x-5)^2 + \left(y - \frac{7}{2}\right)^2 = \left(\frac{5}{2}\right)^2.$$

(You can simplify this to make it look nicer if you like.)

2 *Find the exact value (in radians) of the following angles. (For each one, show at least one intermediate step, such as a diagram, an equation, or a numerical calculation.)*

a $300°$

$$300° = 300 \cdot \frac{\pi}{180} = \frac{5\pi}{3}.$$

b $\tan^{-1}\left(-\sqrt{3}\right)$

Since

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

(which I find by trial and error among 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$), I know that

$$\arctan\sqrt{3} = \frac{\pi}{3}.$$

Therefore,

$$\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}.$$

c $\sin^{-1}\sin\frac{5\pi}{4}$

Since $5\pi/4 > \pi/2$, that is *not* the answer. One possibility is $5\pi/4 - 2\pi = -3\pi/4$, but unfortunately $-3\pi/4 < -\pi/2$. Instead, I'll try

$$\pi - \frac{5\pi}{4} = -\frac{\pi}{4}.$$

Since $-\pi/2 \leq -\pi/4 \leq \pi/2$, this is the answer:

$$\arcsin\sin\frac{5\pi}{4} = -\frac{\pi}{4}.$$

3 *Find the exact value of the following expressions. (For each one, show at least one intermediate step, such as a diagram, an equation, or a numerical calculation.)*

a $\sin\left(-\frac{\pi}{3}\right)$

I know that $\sin(\pi/3) = \sqrt{3}/2$, so

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

b $2\sin\frac{\pi}{3} - 3\cos\frac{\pi}{6}$

I know that $\sin(\pi/3) = \sqrt{3}/2$ and that $\cos(\pi/6) = \sqrt{3}/2$, so

$$2\sin\frac{\pi}{3} - 3\cos\frac{\pi}{6} = 2\left(\frac{\sqrt{3}}{2}\right) - 3\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}.$$

c $\sin\cos^{-1}\frac{3}{5}$

Whatever $\arccos(3/5)$ is, it must be between 0 and π, so its sine is nonnegative. Therefore,

$$\sin\cos^{-1}\frac{3}{5} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}.$$

4 *Consider the function given by the equation*

$$y = 3\sin\left(2x + \frac{\pi}{4}\right).$$

a *What is the period of this function? (Show what numerical calculation you make or what equation you solve.)*

The angular frequency is $\omega = 2$, so the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi.$$

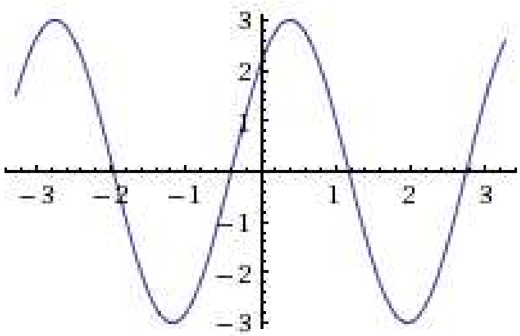
b What is the phase shift of this function? (Show what numerical calculation you make or what equation you solve.)

The angular phase shift is $\phi = -\pi/4$, so the phase shift is

$$\psi = \frac{\phi}{\omega} = \frac{-\pi/4}{2} = -\frac{\pi}{8}.$$

c Sketch a graph of this function, labelling at least three points with at least three different y-coordinates.

Since the phase shift is $-\pi/8$ and the average is 0, one of my points is $(-\pi/8, 0) \approx (-0.4, 0)$. Since the period is π, one fourth of which is $\pi/4$, and the amplitude is 3, another point is $(-\pi/8 + \pi/4, 0 + 3) = (\pi/8, 3) \approx (0.4, 3)$. Since three fourths of the period is $3\pi/4$, yet another point is $(-\pi/8 + 3\pi/4, 0 - 3) = (5\pi/8, -3) \approx (2.0, -3)$. These should be enough to see the graph below (which really should have arrows at the ends):



5 *A circular racetrack has a radius of 75 metres. A racecar on the track is travelling at 75 metres per second. You watch as the car travels through an angle of 60 degrees.*

a How long did it take the car to travel that distance? (Show what numerical calculation you make or what equation you solve.)

The distance travelled by the car was

$$75\,\mathrm{m} \cdot 60^\circ = 75\frac{\pi}{3}\,\mathrm{m} = 25\pi\,\mathrm{m},$$

so it took

$$\frac{25\pi\,\mathrm{m}}{75\,\mathrm{m/s}} = \frac{\pi}{3}\,\mathrm{s}.$$

b There is a wire connected to the car and the centre of the track. How much area is swept out under this wire in this time? (Show what numerical calculation you make or what equation you solve.)

The distance travelled by the car was

$$75\,\mathrm{m} \cdot 60^\circ = 75\frac{\pi}{3}\,\mathrm{m} = 25\pi\,\mathrm{m},$$

so the area swept out by it was

$$\frac{1}{2}(75\,\mathrm{m})(25\pi\,\mathrm{m}) = \frac{1875\pi}{2}\,\mathrm{m}^2.$$

6 Extra credit: *From an airplane* 5000 *feet above the ground, two sightings are taken,* 1 *second apart, of a landmark directly in front of the plane (but on the ground). First, the angle from the vertical to the landmark is* $30°$*; after* 1 *second, this angle is* $15°$*. What is the airplane's speed? (Show what numerical calculation you make or what equation you solve. You may leave an exact answer involving trigonometric operations, or you may use a calculator and round your answer; in either case, be sure to give correct units.)*

At the first sighting, the horizontal distance along the ground from the airplane to the landmark was $5000 \tan 30°$ feet. At the second sighting, that distance was $5000 \tan 15°$ feet. Since it took ony 1 second to travel between these, the speed is

$$5000 \tan 30° - 5000 \tan 15° \approx 1550$$

feet per second. (With 5280 feet in a mile and 3600 seconds in an hour, this is approximately 1050 miles per hour, which is pretty fast but not unheard of.)