Practice Problems

These problems are not to be handed in, but try them first; do as many of them as you need until they're easy, or make up more along the same lines if you need more practice.

Approximate the following integrals using Riemann sums with 8 terms.

1 $\int_1^2 \frac{\mathrm{d}t}{t}$

2 $\int_0^1 \frac{\mathrm{d}x}{x^2+1}$

3 $\int_{-1}^1 \mathrm{e}^{-z^2/2}\,\mathrm{d}z$

4 $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$

Due Problems

These problems are due December 6 Tuesday.

For each of the following integrals approximate it using Riemann sums with 4 terms; use either a lower or upper Riemann sum. **Extra credit:** Do both, and state how precisely you have the true value of the integral.

1 $\int_0^4 \sqrt{x^2+3}\,\mathrm{d}x$

2 $\int_1^2 \frac{x^2}{x+5}\,\mathrm{d}x$

Answers to Practice Problems

Here are the answers to the Practice Problems from the beginning of the assignment.

1 First, a technicality: I take the integrand $\mathrm{d}t/t$, divide by $\mathrm{d}t$ to get $1/t$, and differentiate to get $-1/t^2\,\mathrm{d}t$. Since $-1/t^2$ is negative for t between 1 and 2, I know that $1/t$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of t while the upper Riemann sum will come by using smaller values of t.

If I divide the interval from 1 to 2 into 8 pieces, then each piece has a width of $\frac{2-1}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $t = 1 + 0.125$ to $t = 2$, always rounding down; for the upper Riemann sum, I'll calculate heights form $t = 1$ to $t = 2 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

t	lower Riemann sum	upper Riemann sum
1	(not used)	1
1.125	0.888	0.889
1.25	0.8	0.8
1.375	0.727	0.728
1.5	0.666	0.667
1.625	0.615	0.616
1.75	0.571	0.572
1.875	0.533	0.534
2	0.5	(not used)
total	5.300	5.806

Therefore, the lower Riemann sum is $5.300 \cdot 0.125 \approx 0.662$, while the upper Riemann sum is $5.806 \cdot 0.125 \approx 0.726$.

Therefore, I conclude that

$$\int_1^2 \frac{\mathrm{d}t}{t} \approx 0.7$$

to one decimal place. (As it happens, the correct value is $\ln 2 \approx 0.693$, which a scientific calculator can find quickly and efficiently using a variation of the method above.)

2 First, a technicality: I take the integrand $\mathrm{d}x/(x^2+1)$, divide by $\mathrm{d}x$ to get $1/(x^2+1)$, and differentiate to get $-2x\big/(x^2+1)^2\,\mathrm{d}x$. Since $-2x\big/(x^2+1)^2$ is negative for x between 0 and 1, I know that $1/(x^2+1)$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of x while the upper Riemann sum will come by using smaller values of x.

If I divide the interval from 0 to 1 into 8 pieces, then each piece has a width of $\frac{1-0}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $x = 0 + 0.125$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights form $x = 0$ to $x = 1 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
0	(not used)	1
0.125	0.984	0.985
0.25	0.941	0.942
0.375	0.876	0.877
0.5	0.8	0.8
0.625	0.719	0.720
0.75	0.64	0.64
0.875	0.566	0.567
1	0.5	(not used)
total	6.026	6.531

Therefore, the lower Riemann sum is $6.026 \cdot 0.125 \approx 0.753$, while the upper Riemann sum is $6.531 \cdot 0.125 \approx 0.817$.

Therefore, I conclude that

$$\int_0^1 \frac{\mathrm{d}x}{x^2+1} \approx 0.8$$

to one decimal place. (As it happens, the correct value is $\pi/4 \approx 0.785$, which you can calculate using trigonometry.)

3 First, a technicality: I take the integrand $\mathrm{e}^{-z^2/2}\,\mathrm{d}z$, divide by $\mathrm{d}z$ to get $\mathrm{e}^{-z^2/2}$, and differentiate to get $-z\mathrm{e}^{-z^2/2}\,\mathrm{d}z$. Since $-z\mathrm{e}^{-z^2/2}$ is positive for x between -1 and 0, I know that $\mathrm{e}^{-z^2/2}$ is increasing there; since $-z\mathrm{e}^{-z^2/2}$ is negative for x between 0 and 1, I know that $\mathrm{e}^{-z^2/2}$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using first smaller values of z and then larger values, while the upper Riemann sum will come by using first larger values of z and then small values.

If I divide the interval from -1 to 1 into 8 pieces, then each piece has a width of $\frac{1--1}{8} = 0.25$. For the lower Riemann sum, I'll calculate heights from $x = -1$ to $x = 0 - 0.25$ and then from $x = 0 + 0.25$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights form $x = -1 + 0.25$ to $x = 0$ and then from $x = 0$ (again!) to $x = 1 - 0.25$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.25. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
-1	0.606	(not used)
-0.75	0.754	0.755
-0.5	0.882	0.883
-0.25	0.969	0.970
0	(not used)	1 (used twice)
0.25	0.969	0.970
0.5	0.882	0.883
0.75	0.754	0.755
1	0.606	(not used)
total	6.422	7.216

Therefore, the lower Riemann sum is $6.422 \cdot 0.25 \approx 1.605$, while the upper Riemann sum is $7.216 \cdot 0.25 \approx 1.804$.

Therefore, I conclude that

$$\int_{-1}^{1} \mathrm{e}^{-z^2/2}\,\mathrm{d}z \approx 1.7,$$

give or take one decimal place. (As it happens, the correct value is $\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{2}/2\right) \approx 1.711$, which you can calculate using a statistics calculator.)

4 First, a technicality: I take the integrand $\sqrt{1-x^2}\,\mathrm{d}x$, divide by $\mathrm{d}x$ to get $\sqrt{1-x^2}$, and differentiate to get $-x\sqrt{1-x^2}/(1-x^2)\,\mathrm{d}x$. Since $-x\sqrt{1-x^2}/(1-x^2)$ is negative for x between 0 and 1, I know that $\sqrt{1-x^2}$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of x while the upper Riemann sum will come by using smaller values of x.

If I divide the interval from 0 to 1 into 8 pieces, then each piece has a width of $\frac{1-0}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $x = 0 + 0.125$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights form $x = 0$ to $x = 1 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
0	(not used)	1
0.125	0.992	0.993
0.25	0.968	0.969
0.375	0.927	0.928
0.5	0.866	0.867
0.625	0.780	0.781
0.75	0.661	0.662
0.875	0.484	0.485
1	0	(not used)
total	5.678	6.685

Therefore, the lower Riemann sum is $5.678 \cdot 0.125 \approx 0.709$, while the upper Riemann sum is $6.685 \cdot 0.25 \approx 0.836$.

Therefore, I conclude that

$$\int_{0}^{1} \sqrt{1-x^2}\,\mathrm{d}x \approx 0.8$$

give or take one decimal place. (As it happens, the correct value is $\pi/4 \approx 0.785$, which you can calculate using trigonometry.)