This exam runs from 12:00 to 1:25; if I've written it well, then you shouldn't need the whole time. You may use any books or notes that you have, but you may not communicate with anybody but me. You can come up and talk to me if you have questions, especially about the instructions. Also, you may use a calculator if you wish.

Take your time, and check over your answers. Read the instructions carefully, and be sure to show everything that they ask. You can always show *more* work if you like. If you're unsure of your answer, then explain what you're unsure about, and show your work so that you can get as much partial credit as possible.

Don't forget to put your name on the exam!!!

1 *In each part, you are given an equation relating x and y. Find the derivative of y with respect to x; that is, find an expression for $\mathrm{d}y/\mathrm{d}x$. (Show at least one intermediate step for each.)*

a. $y = 2(x+1)^3$

$$\begin{aligned}\mathrm{d}y &= \mathrm{d}\big(2(x+1)^3\big) = 2\,\mathrm{d}\big((x+1)^3\big) = 2\big(3(x+1)^2\,\mathrm{d}(x+1)\big) = 6(x+1)^2\,\mathrm{d}x;\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 6(x+1)^2.\end{aligned}$$

b. $y = 3\sqrt{x+1}$

$$\begin{aligned}\mathrm{d}y &= \mathrm{d}(3\sqrt{x+1}) = 3\,\mathrm{d}(\sqrt{x+1}) = 3\frac{\sqrt{x+1}\,\mathrm{d}(x+1)}{2(x+1)} = \frac{3\sqrt{x+1}\,\mathrm{d}x}{2(x+1)};\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3\sqrt{x+1}}{2(x+1)}.\end{aligned}$$

c. $3x + 2y = x^3 + y^2$

$$\begin{aligned}\mathrm{d}(3x+2y) &= \mathrm{d}(x^3+y^2);\\ \mathrm{d}(3x) + \mathrm{d}(2y) &= \mathrm{d}(x^3) + \mathrm{d}(y^2);\\ 3\,\mathrm{d}x + 2\,\mathrm{d}y &= 3x^2\,\mathrm{d}x + 2y\,\mathrm{d}y;\\ 2\,\mathrm{d}y - 2y\,\mathrm{d}y &= 3x^2\,\mathrm{d}x - 3\,\mathrm{d}x;\\ (2-2y)\,\mathrm{d}y &= (3x^2-3)\,\mathrm{d}x;\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2-3}{2-2y}.\end{aligned}$$

2 *Find the first and second derivatives of each of the following functions; that is, find formulas for both f' and f''. (Show at least one intermediate step for each.)*

a. $f(x) = 3x^2 + 2$

$$\begin{aligned}\mathrm{d}f(x) &= \mathrm{d}(3x^2+2) = \mathrm{d}(3x^2) = 3\,\mathrm{d}(x^2) = 3\,(2x\,\mathrm{d}x) = 6x\,\mathrm{d}x;\\ f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= 6x;\\ \mathrm{d}f'(x) &= \mathrm{d}(6x) = 6\,\mathrm{d}x;\\ f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= 6.\end{aligned}$$

b. $f(x) = \dfrac{x+3}{x-2}$

$$\begin{aligned}
\mathrm{d}f(x) &= \mathrm{d}\left(\frac{x+3}{x-2}\right) = \frac{(x-2)\,\mathrm{d}(x+3) - (x+3)\,\mathrm{d}(x-2)}{(x-2)^2} = \frac{(x-2)\,\mathrm{d}x - (x+3)\,\mathrm{d}x}{(x-2)^2} = \frac{-5\,\mathrm{d}x}{(x-2)^2};\\
f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \frac{-5}{(x-2)^2};\\
\mathrm{d}f'(x) &= \mathrm{d}\left(\frac{-5}{(x-2)^2}\right) = -\frac{-5\,\mathrm{d}\big((x-2)^2\big)}{\big((x-2)^2\big)^2} = \frac{5\big(2(x-2)\,\mathrm{d}(x-2)\big)}{(x-2)^4} = \frac{10\,\mathrm{d}x}{(x-2)^3};\\
f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= \frac{10}{(x-2)^3}.
\end{aligned}$$

3 *The population of a certain city is given approximately by*

$$C = 100 + 2t^2,$$

where C is the city's population in thousands and t is the time in years since the city was founded. The population of the city's metropolitan area is given approximately by

$$M = C + 0.1tC,$$

where M is the metropolitan population in thousands.

a. *Ten years after the city was founded, how fast is its population growing? (Show at least what numerical calculation you make, and be sure to include appropriate units.)*

$$\begin{aligned}
\mathrm{d}C &= \mathrm{d}(100 + 2t^2) = \mathrm{d}(2t^2) = 2\,\mathrm{d}(t^2) = 2(2t\,\mathrm{d}t) = 4t\,\mathrm{d}t;\\
\frac{\mathrm{d}C}{\mathrm{d}t} &= 4t.
\end{aligned}$$

Ten years after the city was founded, $t = 10$; then

$$\frac{\mathrm{d}C}{\mathrm{d}t} = 4(10) = 40.$$

Therefore, the city's population is growing at a rate of 40 **thousand per year**.

b. *Ten years after the city was founded, how fast is the population of its metropolitan area growing? (Show at least what numerical calculation you make, and be sure to include appropriate units.)*

$$\begin{aligned}
\mathrm{d}M &= \mathrm{d}(C + 0.1tC) = \mathrm{d}C + \mathrm{d}(0.1tC) = \mathrm{d}C + 0.1\,\mathrm{d}(tC)\\
&= \mathrm{d}C + 0.1(C\,\mathrm{d}t + t\,\mathrm{d}C) = (1 + 0.1t)\mathrm{d}C + 0.1C\,\mathrm{d}t;\\
\frac{\mathrm{d}M}{\mathrm{d}t} &= (1 + 0.1t)\frac{\mathrm{d}C}{\mathrm{d}t} + 0.1C.
\end{aligned}$$

When $t = 10$,

$$C = 100 + 2(10)^2 = 300$$

and $\mathrm{d}C/\mathrm{d}t = 40$ (from part a), so

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \big(1 + 0.1(10)\big)(40) + 0.1(300) = 110.$$

Therefore, the metropolitan area's population is growing at a rate of 110 **thousand per year**.

4 *Evaluate the following limits. (Show at least one intermediate step for each.)*

a. $\lim_{x\to 5}\left(\dfrac{x^2-5}{(x-5)^2}\right)$

$$\lim_{x\to 5}\left(\frac{x^2-5}{(x-5)^2}\right)=\frac{(5)^2-5}{(5-5)^2}=\frac{20}{0},$$

which may be infinite. If $x<5$, then $x-5<0$, so $(x-5)^2>0$; if $x>5$, then $x-5>0$, so $(x-5)^2>0$ again. Therefore,

$$\lim_{x\to 5}\left(\frac{x^2-5}{(x-5)^2}\right)=20(\infty)=\infty.$$

b. $\lim_{x\to -\infty}(4-x^2)$

$$\lim_{x\to -\infty}(4-x^2)=4-(-\infty)^2=4-\infty^2=4-\infty=-\infty+4=-\infty.$$

c. $\lim_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right)$

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)=\frac{(3)^2-9}{(3)-3}=\frac{0}{0},$$

which is indeterminate. Applying L'Hôpital's Rule,

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)=\lim_{x\to 3}\left(\frac{\mathrm{d}(x^2-9)}{\mathrm{d}(x-3)}\right)=\lim_{x\to 3}\left(\frac{2x\,\mathrm{d}x}{\mathrm{d}x}\right)=\lim_{x\to 3}(2x)=2(3)=6.$$

Alternatively, using algebra to simplify,

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)=\lim_{x\to 3}\left(\frac{(x-3)(x+3)}{x-3}\right)=\lim_{x\to 3}(x+3)=(3)+3=6.$$

5 *For each of the following functions, find its maximum and minimum value, if they exist. (Show at least what numerical calculation you make for each, as well as at least one other intermediate step.) For* ***extra credit****, sketch a graph of one function, making sure that any maximum or minimum appears on the graph. (Be sure to label the scale on your graph.)*

a. **Extra credit.** $f(x)=\sqrt{900-x^2}$
The function f is undefined if

$$\begin{gathered}0>900-x^2,\\ x^2>900,\\ |x|>30,\\ x<-30 \text{ or } x>30.\end{gathered}$$

So only looking at $-30\le x\le 30$,

$$\begin{aligned}\mathrm{d}f(x)=\mathrm{d}\left(\sqrt{900-x^2}\right)&=\frac{-2x\sqrt{900-x^2}\,\mathrm{d}x}{2(900-x^2)};\\ f'(x)=\frac{\mathrm{d}f(x)}{\mathrm{d}x}&=-\frac{x\sqrt{900-x^2}}{900-x^2}.\end{aligned}$$

This is undefined only if

$$\begin{gathered} 0 = 900 - x^2 = (30 - x)(30 + x), \\ 30 - x = 0 \text{ or } 30 + x = 0, \\ x = \pm 30; \end{gathered}$$

it is zero only if

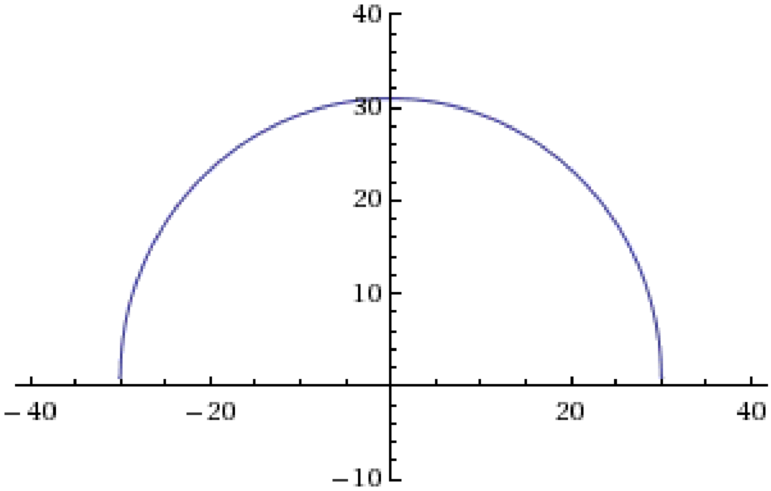
$$\begin{gathered} 0 = x\sqrt{900 - x^2}, \\ 0 = x \text{ or } 0 = 900 - x^2, \\ x = 0 \text{ or } x = \pm 30. \end{gathered}$$

So,

$$\begin{aligned} f(0) &= \sqrt{900 - (0)^2} = \sqrt{900} = 30, \\ f(\pm 30) &= \sqrt{900 - (\pm 30)^2} = 0. \end{aligned}$$

Therefore, the **maximum is** 30 (which occurs at 0), while the **minimum is** 0 (which occurs at both ± 30).

To graph this, I should include both ± 30 in the window horizontally and include both 0 and 30 in the window vertically. Here is a graph with the horizontal window from -40 to 40 and the vertical window from -10 to 40:



b. $f(x) = x^4 + 40x^3 + 400x^2$

The function f is always defined.

$$\begin{aligned} \mathrm{d}f(x) &= \mathrm{d}(x^4 + 40x^3 + 400x^2) = 4x^3\,\mathrm{d}x + 120x^2\,\mathrm{d}x + 800x\,\mathrm{d}x; \\ f'(x) &= \frac{\mathrm{d}f(x)}{\mathrm{d}x} = 4x^3 + 120x^2 + 800x. \end{aligned}$$

This is always defined; it is zero only if

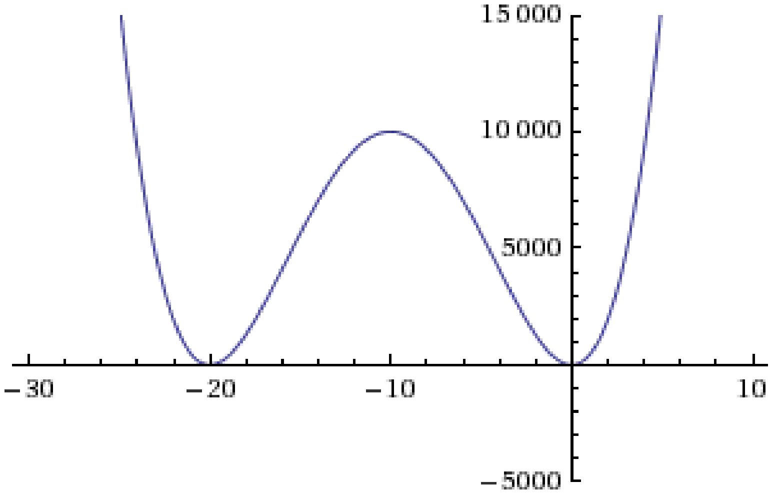
$$\begin{gathered} 0 = 4x^3 + 120x^2 + 800x = 4x^2(x^2 + 30x + 200) = 4x^2(x + 10)(x + 20), \\ 4x^2 = 0 \text{ or } x + 10 = 0 \text{ or } x + 20 = 0, \\ x = 0 \text{ or } x = -10 \text{ or } x = -20. \end{gathered}$$

So,

$$
\begin{aligned}
f(0) &= (0)^4 + 40(0)^3 + 400(0)^2 = 0, \\
f(-10) &= (-10)^4 + 40(-10)^3 + 400(-10)^2 = 10\,000, \\
f(-20) &= (-20)^4 + 40(-20)^3 + 400(-20)^2 = 0, \\
\lim_{x \to \infty} (x^4 + 40x^3 + 400x^2) &= (\infty)^4 + 40(\infty)^3 + 400(\infty)^2 = \infty, \\
\lim_{x \to -\infty} (x^4 + 40x^3 + 400x^2) &= \infty
\end{aligned}
$$

(the last of which was given on the board). Therefore, there is **no maximum**, while the **minimum is** 0 (which occurs at both 0 and -20).

To graph this, I should include both -20 and 0 in the window horizontally and include both 0 and 10 000 in the window vertically. Here is a graph with the horizontal window from -30 to 10 and the vertical window from $-5\,000$ to 15 000:



6 *Suppose that the revenue from selling x specialty items in a year is*

$$R = 10x - x^2,$$

while the cost is

$$C = 4x + 100,$$

both measured in thousands of dollars.

a. *How many items should be sold in a year to maximise revenue? (Show at least what numerical calculation you make.)*

$$\mathrm{d}R = \mathrm{d}(10x - x^2) = 10\,\mathrm{d}x - 2x\,\mathrm{d}x.$$

While $\mathrm{d}x$ varies freely, this is zero only if

$$
\begin{aligned}
0 &= 10 - 2x, \\
x &= 5.
\end{aligned}
$$

This is the number of items that should be sold to maximise revenue.

b. *How many should be sold to maximise profit? (Show at least what numerical calculation you make.)*

If P is profit, then

$$\mathrm{d}P = \mathrm{d}(R - C) = \mathrm{d}R - \mathrm{d}C = 10\,\mathrm{d}x - 2x\,\mathrm{d}x - 4\,\mathrm{d}x.$$

While $\mathrm{d}x$ varies freely, this is zero only if

$$\begin{aligned} 0 &= 10 - 2x - 4, \\ x &= 3. \end{aligned}$$

This is the number of items that should be sold to maximise profit.

c. **Extra credit.** *How many should be sold to minimise cost? (Show at least what numerical calculation you make.)*

While $\mathrm{d}x$ varies freely, $\mathrm{d}C$ is never zero. However, since $\mathrm{d}C/\mathrm{d}x = 4$ is always positive, we can minimise cost by minimising x, and it doesn't make sense for x to go below zero. Therefore, **no items should be sold** to minimise cost.