Practice Problems

The first few problems are for practice; do as many of them as you need until they're easy, or make up more for yourself along the same lines. (The answers appear at the end of this assignment.)

Approximate the following integrals using Riemann sums with 8 terms.

1 $\int_1^2 \frac{\mathrm{d}t}{t}$

2 $\int_0^1 \frac{\mathrm{d}x}{x^2+1}$

3 $\int_{-1}^1 \mathrm{e}^{-z^2/2}\,\mathrm{d}z$

4 $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$

Due Problems

The following problems are due June 9 Thursday.

Approximate the following integrals using Riemann sums with 4 terms. Use either a lower or upper Riemann sum; don't bother to do both.

1 $\int_0^4 \sqrt{x^2+3}\,\mathrm{d}x$

2 $\int_1^2 \frac{x+5}{x^2}\,\mathrm{d}x$

Answers to Practice Problems

Here are the answers to the Practice Problems from the beginning of the assignment.

1 First, a technicality: I take the integrand $\mathrm{d}t/t$, divide by $\mathrm{d}t$ to get $1/t$, and differentiate to get $-1/t^2\,\mathrm{d}t$. Since $-1/t^2$ is negative for t between 1 and 2, I know that $1/t$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of t while the upper Riemann sum will come by using smaller values of t.

If I divide the interval from 1 to 2 into 8 pieces, then each piece has a width of $\frac{2-1}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $t = 1 + 0.125$ to $t = 2$, always rounding down; for the upper Riemann sum, I'll calculate heights form $t = 1$ to $t = 2 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

t	lower Riemann sum	upper Riemann sum
1	(not used)	1
1.125	0.888	0.889
1.25	0.8	0.8
1.375	0.727	0.728
1.5	0.666	0.667
1.625	0.615	0.616
1.75	0.571	0.572
1.875	0.533	0.534
2	0.5	(not used)
total	5.300	5.806

Therefore, the lower Riemann sum is $5.300 \cdot 0.125 \approx 0.662$, while the upper Riemann sum is $5.806 \cdot 0.125 \approx 0.726$.

Therefore, I conclude that

$$\int_1^2 \frac{\mathrm{d}t}{t} \approx 0.7$$

to one decimal place. (As it happens, the correct value is $\ln 2 \approx 0.693$, which a scientific calculator can find quickly and efficiently using a variation of the method above.)

2 First, a technicality: I take the integrand $\mathrm{d}x/(x^2+1)$, divide by $\mathrm{d}x$ to get $1/(x^2+1)$, and differentiate to get $-2x\big/(x^2+1)^2\,\mathrm{d}x$. Since $-2x\big/(x^2+1)^2$ is negative for x between 0 and 1, I know that $1/(x^2+1)$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of x while the upper Riemann sum will come by using smaller values of x.

If I divide the interval from 0 to 1 into 8 pieces, then each piece has a width of $\frac{1-0}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $x = 0 + 0.125$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights form $x = 0$ to $x = 1 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
0	(not used)	1
0.125	0.984	0.985
0.25	0.941	0.942
0.375	0.876	0.877
0.5	0.8	0.8
0.625	0.719	0.720
0.75	0.64	0.64
0.875	0.566	0.567
1	0.5	(not used)
total	6.026	6.531

Therefore, the lower Riemann sum is $6.026 \cdot 0.125 \approx 0.753$, while the upper Riemann sum is $6.531 \cdot 0.125 \approx 0.817$.

Therefore, I conclude that

$$\int_0^1 \frac{dx}{x^2 + 1} \approx 0.8$$

to one decimal place. (As it happens, the correct value is $\pi/4 \approx 0.785$, which you can calculate using trigonometry.)

3 First, a technicality: I take the integrand $e^{-z^2/2}\,dz$, divide by dz to get $e^{-z^2/2}$, and differentiate to get $-ze^{-z^2/2}\,dz$. Since $-ze^{-z^2/2}$ is positive for x between -1 and 0, I know that $e^{-z^2/2}$ is increasing there; since $-ze^{-z^2/2}$ is negative for x between 0 and 1, I know that $e^{-z^2/2}$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using first smaller values of z and then larger values, while the upper Riemann sum will come by using first larger values of z and then small values.

If I divide the interval from 0 to 1 into 8 pieces, then each piece has a width of $\frac{1--1}{8} = 0.25$. For the lower Riemann sum, I'll calculate heights from $x = -1$ to $x = 0 - 0.25$ and then from $x = 0 + 0.25$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights form $x = -1 + 0.25$ to $x = 0$ and then from $x = 0$ (again!) to $x = 1 - 0.25$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.25. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
-1	0.606	(not used)
-0.75	0.754	0.755
-0.5	0.882	0.883
-0.25	0.969	0.970
0	(not used)	1 (used twice)
0.25	0.969	0.970
0.5	0.882	0.883
0.75	0.754	0.755
1	0.606	(not used)
total	6.422	7.216

Therefore, the lower Riemann sum is $6.422 \cdot 0.25 \approx 1.605$, while the upper Riemann sum is $7.216 \cdot 0.25 \approx 1.804$.

Therefore, I conclude that

$$\int_{-1}^{1} \mathrm{e}^{-z^2/2}\,\mathrm{d}z \approx 1.7,$$

give or take one decimal place. (As it happens, the correct value is $\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{2}/2\right) \approx 1.711$, which you can calculate using a statistics calculator.)

4 First, a technicality: I take the integrand $\sqrt{1-x^2}\,\mathrm{d}x$, divide by $\mathrm{d}x$ to get $\sqrt{1-x^2}$, and differentiate to get $-2x\sqrt{1-x^2}\,\mathrm{d}x/2(1-x^2)$. Since $-2x\sqrt{1-x^2}\,\mathrm{d}x/2(1-x^2)$ is negative for x between 0 and 1, I know that $\sqrt{1-x^2}$ is decreasing there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of x while the upper Riemann sum will come by using smaller values of x.

If I divide the interval from 0 to 1 into 8 pieces, then each piece has a width of $\frac{1-0}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $x = 0 + 0.125$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights form $x = 0$ to $x = 1 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
0	(not used)	1
0.125	0.992	0.993
0.25	0.968	0.969
0.375	0.927	0.928
0.5	0.866	0.867
0.625	0.780	0.781
0.75	0.661	0.662
0.875	0.484	0.485
1	0	(not used)
total	5.678	6.685

Therefore, the lower Riemann sum is $5.678 \cdot 0.125 \approx 0.709$, while the upper Riemann sum is $6.685 \cdot 0.25 \approx 0.836$.

Therefore, I conclude that

$$\int_{0}^{1} \sqrt{1-x^2}\,\mathrm{d}x \approx 0.8$$

give or take one decimal place. (As it happens, the correct value is $\pi/4 \approx 0.785$, which you can calculate using trigonometry.)