Many of these problems are based on exercises from the official textbook; I have indicated which (by chapter number, section number, and exercise number) when this occurs.

Practice Problems

The first few problems are for practice; do as many of them as you need until they're easy, or make up more for yourself along the same lines.

1 *(1.5.83) The median weight for a baby boy in the United States, whose age is t months, is approximately $8.15 + 1.82t - 0.0596\,t^2 + 0.000\,758\,t^3$ pounds. How fast is this median weight changing for a hypothetical 10-month-old baby?*

Let w be the predicted weight in pounds, so I have an equation relating w and t. (You could do this problem without ever introducing the variable w.) I differentiate this equation, divide both sides by $\mathrm{d}t$ to get the rate of change, and use the given value of $t = 10$.

$$\begin{aligned}
w &= 8.15 + 1.82t - 0.0596t^2 + 0.000\,758t^3;\\
\mathrm{d}w &= 1.82\,\mathrm{d}t - 0.0596(2t\,\mathrm{d}t) + 0.000\,758(3t^2\,\mathrm{d}t);\\
\mathrm{d}w &= (1.82 - 0.1192t + 0.002\,274t^2)\,\mathrm{d}t;\\
\frac{\mathrm{d}w}{\mathrm{d}t} &= 1.82 - 0.1192t + 0.002\,274t^2;\\
\frac{\mathrm{d}w}{\mathrm{d}t} &= 1.82 - 0.1192(10) + 0.002\,274(10)^2;\\
\frac{\mathrm{d}w}{\mathrm{d}t} &= 0.8554.
\end{aligned}$$

Since w is in pounds and t is in months, the median weight is increasing by approximately 0.86 pounds per month.

2 *(1.5.81, 2.7.40) If r is the radius of a circular burn and A is its area, then $A = \pi r^2$ (by the geometric formula for area of a circle), where $\pi \approx 3.14$ is a constant. At a certain moment, the radius is 3 centimetres but is shrinking at the rate of 1 centimetre per day; how is the area changing at that moment?*

First, I differentiate the equation relating area to radius. Next, since the problem discusses the rate of change with time, I divide by $\mathrm{d}t$ (where t stands for time). Finally, I use the given values of r and $\mathrm{d}r/\mathrm{d}t$ to find $\mathrm{d}A/\mathrm{d}t$.

$$\begin{aligned}
A &= \pi r^2;\\
\mathrm{d}A &= \pi(2r\,\mathrm{d}r);\\
\mathrm{d}A &= 2\pi r\,\mathrm{d}r;\\
\frac{\mathrm{d}A}{\mathrm{d}t} &= 2\pi r\frac{\mathrm{d}r}{\mathrm{d}t};\\
\frac{\mathrm{d}A}{\mathrm{d}t} &= 2\pi(3\,\mathrm{cm})(-1\,\mathrm{cm/dy});\\
\frac{\mathrm{d}A}{\mathrm{d}t} &= -6\pi\,\mathrm{cm}^2/\mathrm{dy}.
\end{aligned}$$

Therefore, the area is decreasing by 6π square centimetres per day (which is approximately $18.9\,\mathrm{cm}^2/\mathrm{dy}$).

3 *(2.5.29) The marketing department of a business determines that the number of items that can be sold in a day is $300 - 2p$ when each item is sold for a price of p dollars. The engineering department estimates that the cost of producing x items per day is $4000 + 0.25x^2$ dollars per day on average. To make the largest amount of profit, what should the price be, how many items should be produced (and sold) in an average day, and how much profit should the business expect?*

If p is the price (in dollars) of an item and x is the number of items sold in a day, then $x = 300 - 2p$. If R is the revenue (in dollars per day), then $R = xp$. If C is the cost (in dollars per day) of producing x items per day, then $C = 4000 + 0.25x^2$. If P is the profit (in dollars per day), then $P = R - C$. (Note that p and P are different!)

One way to solve the problem is to put everything in terms of the unit price p, but it will be simpler only to work out the differentials. I have:

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}(300 - 2p) = -2\,\mathrm{d}p; \\ \mathrm{d}R &= \mathrm{d}(xp) = p\,\mathrm{d}x + x\,\mathrm{d}p = p(-2\,\mathrm{d}p) + x\,\mathrm{d}p = (x - 2p)\,\mathrm{d}p; \\ \mathrm{d}C &= \mathrm{d}(4000 + 0.25x^2) = 0.25(2x\,\mathrm{d}x) = 0.5x(-2\,\mathrm{d}p) = -1x\,\mathrm{d}p; \\ \mathrm{d}P &= \mathrm{d}(R - C) = \mathrm{d}R - \mathrm{d}C = (x - 2p)\,\mathrm{d}p - (-x)\mathrm{d}p = (2x - 2p)\,\mathrm{d}p. \end{aligned}$$

At the maximum profit, $\mathrm{d}P$ must be zero (or undefined, which never happens), so $2x - 2p = 0$. Now I put x in terms of p to solve:

$$\begin{aligned} 2x - 2p &= 0; \\ 2(300 - 2p) - 2p &= 0; \\ 600 - 6p &= 0; \\ p &= 100. \end{aligned}$$

If $p = 100$, then $x = 300 - 2(100) = 100$, $R = (100)(100) = 10\,000$, $C = 4000 + 0.25(100)^2 = 6500$, and $P = 10\,000 - 6500 = 3500$. Therefore, the price on one item should be \$100, there should be 100 items produced per day, and the business can expect a profit of \$3500 per day. (Notice that if $p = 0$, then $R = 0$ but $C > 0$ so $P < 0$. Similarly, if $p = 150$, then $x = 0$, so again $R = 0$ but $C > 0$ so $P < 0$. These extreme cases tell me that I have really calculated a *maximum* profit.)

4 *(2.5.13) A lifeguard wants to rope off a rectangular swimming area in front of a beach, using 180 yards of floaty rope. (There is no rope along the beach itself.) What is the largest area that the lifeguard can enclose?*

Let x be the length of the rectangle along the beach, and let y be the length perpendicular to the beach. Then the total length of rope is $x + 2y$, so $x + 2y = 180\,\mathrm{yd}$. As for the area, this is simply $A = xy$. As we vary x and y, $\mathrm{d}x + 2\,\mathrm{d}y = 0$, so $\mathrm{d}x = -2\,\mathrm{d}y$. Therefore,

$$\mathrm{d}A = y\,\mathrm{d}x + x\,\mathrm{d}y = y(-2\,\mathrm{d}y) + x\,\mathrm{d}y = (x - 2y)\,\mathrm{d}y.$$

For the maximum area, $\mathrm{d}A = 0$, so $x - 2y = 0$.

Since $x + 2y = 180\,\mathrm{yd}$ and $x - 2y = 0$, I find that $x = 2y$, so $4y = 180\,\mathrm{yd}$, so $y = 45\,\mathrm{yd}$. Then $x = 2(45\,\mathrm{yd}) = 90\,\mathrm{yd}$, and $A = (90\,\mathrm{yd})(45\,\mathrm{yd}) = 405\,\mathrm{sq\,yd}$. Therefore, the maximum possible area is 405 square yards. (Notice that if $y = 0$, then $A = 0$, while if $y = 90\,\mathrm{yd}$, then $x = 0$, so again $A = 0$. These extreme cases tell me that I have really calculated a *maximum* area.)

Due Problems

The following problems were due April 19 Tuesday.

1 *(2.3.62) A patient is given an injection of medication. Suppose that, t hours after the injection, the amount of medication (in cubic centimetres) in the bloodstream of the patient is $\frac{100}{t^2+1}$.*

Let V be the amount of medication after t hours, in cubic centimetres. That is,

$$V := \frac{100}{t^2+1}.$$

I accidentally gave $V = 100t^2 + 1$ on the hand-out, but the question isn't realistic with that formula. So I'll give answers for both the intended formula and the mistake.

a. *How much medication is in the patient's bloodstream after 1 hour? (Show at least what numerical calculation you make.)*

After 1 hour, $t = 1$, so

$$V = \frac{100}{(1)^2 + 1} = 50.$$

Therefore, there are $50\,\mathrm{cm}^3$ of medication in the blood at that time. (In medicine, this often written 50 cc or 50 mL; the latter stands for 50 millilitres, which is the same volume.)

Using the mistaken formula instead,

$$V = 100(1)^2 + 1 = 101,$$

so in that case there would be $101\,\mathrm{cm}^3$ in the blood.

b. *How fast is the medication leaving the bloodstream at that time? (Show at least what numerical calculation you make.)*

The speed at which a quantity changes is its derivative with respect to time. So,

$$\begin{aligned} \mathrm{d}V &= \mathrm{d}\left(\frac{100}{t^2+1}\right) \\ &= -\frac{100\,\mathrm{d}(t^2+1)}{(t^2+1)^2} \\ &= -\frac{200t\,\mathrm{d}t}{(t^2+1)^2}; \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= -\frac{200t}{(t^2+1)^2}. \end{aligned}$$

Again, after 1 hour, $t = 1$, so

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{200(1)}{\left((1)^2+1\right)^2} = -50.$$

Therefore, the amount of medicine is changing by $-50\,\mathrm{cm}^3/\mathrm{h}$; in other words, it's leaving the bloodstream at a speed of 50 cubic centimetres per hour.

Using the mistaken formula instead,

$$\begin{aligned} \mathrm{d}V &= \mathrm{d}(100t^2 + 1) \\ &= 200t\,\mathrm{d}t; \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= 200t. \end{aligned}$$

When $t = 1$,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 200(1) = 200.$$

Therefore, the amount of medicine would be changing by $200\,\mathrm{cm}^3/\mathrm{h}$; in other words, it would be entering the bloodstream at a speed of 200 cubic centimetres per hour. (The reason that this formula is unrealistic is that, some time after the injection, the medicine should be *leaving* the bloodstream, not *entering* it.)

2 *Suppose that research for a small automobile company suggests that the annual revenue from selling x cars per year will be $25\,000x - 5x^2$ dollars, while the annual cost of producing x cars per year will be $10\,000 + 5000x$ dollars.*

Let R be the annual revenue, and let C be the annual cost, both in dollars. That is,

$$\begin{aligned} R &:= 25\,000x - 5x^2; \\ C &:= 10\,000 + 5000x. \end{aligned}$$

a. *If the marketing department tries to maximise revenue, what goal will they set as the number of cars to sell in a year? (Show at least what equation you solve to find this.)*

The revenue can only be maximised when its differential is zero:

$$\begin{aligned} R &= 25\,000x - 5x^2; \\ 0 = \mathrm{d}R &= 25\,000\,\mathrm{d}x - 5\,\mathrm{d}(x^2) \\ &= 25\,000\,\mathrm{d}x - 10x\,\mathrm{d}x \\ &= (25\,000 - 10x)\,\mathrm{d}x; \\ 0 &= 25\,000 - 10x; \\ x &= 2500. \end{aligned}$$

Therefore, the marketing department will try to sell 2500 cars per year. (I can divide by $\mathrm{d}x$ because, when setting the marketing department's quota, we are trying to change the number of items sold, so $\mathrm{d}x \neq 0$.)

b. *How many cars should actually be manufactured and sold in a year in order to maximise profit for the company? (Show at least what equation you solve to find this.)*

Let P be annual profit (in dollars), so $P = R - C$. Now I want the differential of P to be zero:

$$\begin{aligned} P &= R - C; \\ 0 = \mathrm{d}P &= \mathrm{d}(25\,000x - 5x^2) - \mathrm{d}(10\,000 + 5000x) \\ &= (25\,000\,\mathrm{d}x - 10x\,\mathrm{d}x) - (5000\,\mathrm{d}x); \\ 0 &= 25\,000 - 10x - 5000; \\ x &= 2000. \end{aligned}$$

Therefore, the company makes the most profit if it sells 2000 cars per year. (So it's best if the marketing department *doesn't* quite meet its quota!)