## Practice Problems

The first few problems are for practice; do as many of them as you need until they're easy, or make up more for yourself along the same lines.

**1** *Find the second derivative of $y$ with respect to $x$:*

a. $y = 5x^3 - 4x^2 + 3x$

Following problem (3.a) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^2 - 8x + 3.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}(15x^2 - 8x + 3) = 30x\,\mathrm{d}x - 8\,\mathrm{d}x,$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = 30x - 8.$$

b. $y = \dfrac{12}{x+5} - 10$

Following problem (3.b) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{12}{(x+5)^2}.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}\left(-\frac{12}{(x+5)^2}\right) = \frac{12\,\mathrm{d}\big((x+5)^2\big)}{\big((x+5)^2\big)^2} = \frac{24(x+5)\,\mathrm{d}(x+5)}{(x+5)^4} = \frac{24\,\mathrm{d}x}{(x+5)^3},$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = \frac{24}{(x+5)^3}.$$

c. $x^2 + y^2 = 1$

Following problem (3.c) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}\left(-\frac{x}{y}\right) = -\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{y^2} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{y^2},$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = \frac{x\,\mathrm{d}y/\mathrm{d}x - y}{y^2} = \frac{x(-x/y) - y}{y^2} = \frac{-x^2 - y^2}{y^3} = -\frac{x^2 + y^2}{y^3}.$$

d. $(x+y)^2 = 1$

Following problem (3.d) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}(-1) = 0,$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = 0.$$

**2** *Find formulas for $f'$ and $f''$:*

a. $f(x) = 3x^3$

I differentiate the formula for $f(x)$ and divide by $\mathrm{d}x$:

$$\begin{aligned} f(x) &= 3x^3; \\ \mathrm{d}f(x) &= \mathrm{d}(3x^3) = 9x^2\,\mathrm{d}x; \\ f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= 9x^2. \end{aligned}$$

To find $f''$, I do this again:

$$\begin{aligned} f'(x) &= 9x^2; \\ \mathrm{d}f'(x) &= \mathrm{d}(9x^2) = 18x\,\mathrm{d}x; \\ f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= 18x. \end{aligned}$$

b. $f(t) = \sqrt{t-5}$

I differentiate the formula for $f(t)$ and divide by $\mathrm{d}t$:

$$\begin{aligned} f(t) &= \sqrt{t-5}; \\ \mathrm{d}f(t) &= \mathrm{d}\big(\sqrt{t-5}\big) = \frac{\sqrt{t-5}\,\mathrm{d}(t-5)}{2(t-5)} = \frac{\sqrt{t-5}\,\mathrm{d}t}{2t-10}; \\ f'(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} &= \frac{\sqrt{t-5}}{2t-10}. \end{aligned}$$

To find $f''$, I do this again:

$$\begin{aligned} f'(t) &= \frac{\sqrt{t-5}}{2t-10}; \\ \mathrm{d}f'(t) &= \mathrm{d}\left(\frac{\sqrt{t-5}}{2t-10}\right) = \frac{(2t-10)\,\mathrm{d}(\sqrt{t-5}) - \sqrt{t-5}\,\mathrm{d}(2t-10)}{(2t-10)^2} \\ &= \frac{\sqrt{t-5}\,\mathrm{d}t - 2\sqrt{t-5}\,\mathrm{d}t}{(2t-10)^2} = -\frac{\sqrt{t-5}\,\mathrm{d}t}{(2t-10)^2}; \\ f''(t) = \frac{\mathrm{d}f'(t)}{\mathrm{d}t} &= -\frac{\sqrt{t-5}}{(2t-10)^2}. \end{aligned}$$

c. $f(x) = \dfrac{x^2}{x-1}$

I differentiate the formula for $f(x)$ and divide by $\mathrm{d}x$:

$$\begin{aligned} f(x) &= \frac{x^2}{x-1}; \\ \mathrm{d}f(x) &= \mathrm{d}\left(\frac{x^2}{x-1}\right) = \frac{(x-1)\,\mathrm{d}(x^2) - x^2\,\mathrm{d}(x-1)}{(x-1)^2} = \frac{2x(x-1)\,\mathrm{d}x - x^2\,\mathrm{d}x}{(x-1)^2}; \\ f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}. \end{aligned}$$

To find $f''$, I do this again:

$$\begin{aligned} f'(x) &= \frac{x^2 - 2x}{(x-1)^2}; \\ \mathrm{d}f'(x) &= \mathrm{d}\left(\frac{x^2 - 2x}{(x-1)^2}\right) = \frac{(x-1)^2\,\mathrm{d}(x^2 - 2x) - (x^2 - 2x)\,\mathrm{d}\big((x-1)^2\big)}{\big((x-1)^2\big)^2} \\ &= \frac{(x-1)^2(2x\,\mathrm{d}x - 2\,\mathrm{d}x) - 2(x^2 - 2x)(x-1)\,\mathrm{d}x}{(x-1)^4}; \\ f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= \frac{(x-1)^2(2x-2) - 2(x^2-2x)(x-1)}{(x-1)^4}. \end{aligned}$$

(This could be simplified further, if you wish.)

d. $f(p) = 5 - p^2$

I differentiate the formula for $f(p)$ and divide by $\mathrm{d}p$:

$$\begin{aligned} f(p) &= 5 - p^2; \\ \mathrm{d}f(p) &= \mathrm{d}(5 - p^2) = -2p\,\mathrm{d}p; \\ f'(p) = \frac{\mathrm{d}f(p)}{\mathrm{d}p} &= -2p. \end{aligned}$$

To find $f''$, I do this again:

$$\begin{aligned} f'(p) &= -2p; \\ \mathrm{d}f'(x) &= \mathrm{d}(-2p) = -2\,\mathrm{d}p; \\ f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= -2. \end{aligned}$$

**Due Problems**

The following problems were due April 26 Tuesday.

**1** *Given that $y = 3x^4 - 5x^2$ always, find the first and second derivatives of $y$ with respect to $x$. (Show at least one intermediate step for each.)*

$$\begin{aligned} y &= 3x^4 - 5x^2; \\ \mathrm{d}y &= 12x^3\,\mathrm{d}x - 10x\,\mathrm{d}x; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 12x^3 - 10x; \\ \mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 36x^2\,\mathrm{d}x - 10\,\mathrm{d}x; \\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y &= 36x^2 - 10. \end{aligned}$$

**2** *Given that $f(x) = \sqrt{x} + 1/x$ always, find the first and second derivatives of the function $f$. (Show at least one intermediate step for each.)*

$$\begin{aligned}
f(x) &= \sqrt{x} + \frac{1}{x}; \\
\mathrm{d}f(x) &= \frac{\sqrt{x}\,\mathrm{d}x}{2x} - \frac{1\,\mathrm{d}x}{x^2}; \\
f'(x) &= \frac{\sqrt{x}}{2x} - \frac{1}{x^2}; \\
\mathrm{d}f'(x) &= \frac{2x\,\mathrm{d}(\sqrt{x}) - \sqrt{x}\,\mathrm{d}(2x)}{(2x)^2} + \frac{1\,\mathrm{d}(x^2)}{(x^2)^2} = \frac{\sqrt{x}\,\mathrm{d}x - 2\sqrt{x}\,\mathrm{d}x}{4x^2} + \frac{2x\,\mathrm{d}x}{x^4} = -\frac{\sqrt{x}\,\mathrm{d}x}{4x^2} + \frac{2\,\mathrm{d}x}{x^3}; \\
f''(x) &= -\frac{\sqrt{x}}{4x^2} + \frac{2}{x^3}.
\end{aligned}$$

(You didn't really have to simplify the formula for $f''$ this much.)

**3** **Extra credit.** *In Physics classes, one learns the formula*

$$h = a + vt - \frac{1}{2}gt^2$$

*for the height $h$ of a projectile that is released at a height $a$ with upward velocity $v$, where $t$ is the amount of time after the projectile is released and $g$ is the local downward acceleration of gravity. (Note that $a$, $v$, and $g$ are constants.) Verify that this formula is correct (showing in each case what calculation you must make to verify this), as follows:*

a. *At the time at which the projectile is released ($t = 0$), check that the height really is $a$.*

When $t = 0$,

$$h = a + vt - \frac{1}{2}gt^2 = a + v(0) - \frac{1}{2}g(0)^2 = a,$$

so the initial height really is $a$.

b. *At the time at which the projectile is released, check that the upward speed really is $v$. (Hint: Upward speed is the derivative of height with respect to time.)*

First,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}\left(a + vt - \frac{1}{2}gt^2\right)}{\mathrm{d}t} = \frac{0 + v\,\mathrm{d}t - \frac{1}{2}g(2t\,\mathrm{d}t)}{\mathrm{d}t} = v - gt.$$

When $t = 0$,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = v - g(0) = v,$$

so the initial upward velocity really is $v$.

c. *At all times, check that the downward acceleration of the particle really is $g$. (Hint: Upward acceleration is the second derivative of height with respect to time, and downward acceleration is simply the opposite of upward acceleration.)*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 h = \frac{\mathrm{d}(v - gt)}{\mathrm{d}t} = \frac{0 - g\,\mathrm{d}t}{\mathrm{d}t} = -g,$$

so

$$-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 h = g;$$

that is, the downard acceleration is always $g$.