Practice Problems

The first few problems are for practice; do as many of them as you need until they're easy, or make up more for yourself along the same lines.

1 *Differentiate the following expressions.*

a. 3^{x^2+1}

I can use the rule for a constant base if I remember it:

$$\mathrm{d}\left(3^{x^2+1}\right) = 3^{x^2+1}\ln 3\,\mathrm{d}(x^2+1) = 3^{x^2+1}\ln 3\,(2x\,\mathrm{d}x) = 2x3^{x^2+1}\ln 3\,\mathrm{d}x.$$

Or I can turn everything into expressions with base e, using a simpler differentiation rule but more algebra:

$$\begin{aligned}\mathrm{d}\left(3^{x^2+1}\right) &= \mathrm{d}\left(\mathrm{e}^{(x^2+1)\ln 3}\right) = \mathrm{e}^{(x^2+1)\ln 3}\,\mathrm{d}\big((x^2+1)\ln 3\big) \\ &= 3^{x^2+1}\ln 3\,\mathrm{d}(x^2+1) = 3^{x^2+1}\ln 3\,(2x\,\mathrm{d}x) = 2x3^{x^2+1}\ln 3\,\mathrm{d}x.\end{aligned}$$

b. $\ln(xy+2)$

Since the base is already e, there's really only one way to do this:

$$\mathrm{d}(\ln(xy+2)) = \frac{\mathrm{d}(xy+2)}{xy+2} = \frac{y\,\mathrm{d}x + x\,\mathrm{d}y}{xy+2}.$$

c. $(x+2)^{3x}$

Since the base and exponent are both variable, I'll turn everything into expressions with base e:

$$\begin{aligned}\mathrm{d}\big((x+2)^{3x}\big) &= \mathrm{d}\big(\mathrm{e}^{3x\ln(x+2)}\big) = \mathrm{e}^{3x\ln(x+2)}\,\mathrm{d}(3x\ln(x+2)) \\ &= (x+2)^{3x}\Big(3\ln(x+2)\,\mathrm{d}x + 3x\mathrm{d}\big(\ln(x+2)\big)\Big) \\ &= (x+2)^{3x}\left(3\ln(x+2)\,\mathrm{d}x + 3x\frac{\mathrm{d}(x+2)}{x+2}\right) \\ &= (x+2)^{3x}\left(3\ln(x+2)\,\mathrm{d}x + 3x\frac{\mathrm{d}x}{x+2}\right).\end{aligned}$$

(It would also be possible to do this with one complicated rule.)

d. $\log_5(p+q)$

As in part (a), I can use the rule for a constant base if I remember it:

$$\mathrm{d}(\log_5(p+q)) = \frac{\mathrm{d}(p+q)}{(p+q)\ln 5} = \frac{\mathrm{d}p + \mathrm{d}q}{(p+q)\ln 5}.$$

Or I can turn everything into expressions with base e, using a simpler differentiation rule but more algebra:

$$\mathrm{d}(\log_5(p+q)) = \mathrm{d}\left(\frac{\ln(p+q)}{\ln 5}\right) = \frac{\mathrm{d}(\ln(p+q))}{\ln 5} = \frac{\frac{\mathrm{d}(p+q)}{p+q}}{\ln 5} = \frac{\mathrm{d}p+\mathrm{d}q}{(p+q)\ln 5}.$$

e. $5x^2\mathrm{e}^{2x-7}$

I use the Product Rule first, then tackle the exponential part of the expression:

$$\begin{aligned}\mathrm{d}(5x^2\mathrm{e}^{2x-7}) &= \mathrm{e}^{2x-7}\,\mathrm{d}(5x^2) + 5x^2\,\mathrm{d}(\mathrm{e}^{2x-7}) = \mathrm{e}^{2x-7}(10x\,\mathrm{d}x) + 5x^2\mathrm{e}^{2x-7}\,\mathrm{d}(2x-7) \\ &= 10x\mathrm{e}^{2x-7}\,\mathrm{d}x + 5x^2\mathrm{e}^{2x-7}(2\,\mathrm{d}x) = 10x\mathrm{e}^{2x-7}\,\mathrm{d}x + 10x^2\mathrm{e}^{2x-7}\,\mathrm{d}x.\end{aligned}$$

f. $\log_x(x+1)$

As in part (c), I'll turn everything into expressions with base e:

$$\mathrm{d}\big(\log_x(x+1)\big) = \mathrm{d}\left(\frac{\ln(x+1)}{\ln x}\right) = \frac{\ln x\,\mathrm{d}(\ln(x+1)) - \ln(x+1)\,\mathrm{d}(\ln x)}{(\ln x)^2}$$
$$= \frac{\ln x\,\frac{\mathrm{d}(x+1)}{x+1} - \ln(x+1)\,\frac{\mathrm{d}x}{x}}{(\ln x)^2} = \frac{\ln x\,\frac{\mathrm{d}x}{x+1} - \ln(x+1)\,\frac{\mathrm{d}x}{x}}{(\ln x)^2}.$$

2 *Find the following limits.*

a. $\lim\limits_{x\to 0^+}(x^x)$

First, I'll try plugging in the limiting value:

$$\lim_{x\to 0^+}(x^x) = 0^0,$$

but this is indeterminate. So I'll turn everything into expressions with base e and see where that gets me:

$$\lim_{x\to 0^+}(x^x) = \lim_{x\to 0^+}\mathrm{e}^{x\ln x} = \lim_{x\to 0^+}\mathrm{e}^{\frac{\ln x}{1/x}} = \mathrm{e}^{\frac{-\infty}{\infty}}.$$

This is a case for L'Hôpital in the exponent:

$$\lim_{x\to 0^+}(x^x) = \lim_{x\to 0^+}\mathrm{e}^{\frac{\ln x}{1/x}} = \lim_{x\to 0^+}\mathrm{e}^{\frac{\mathrm{d}(\ln x)}{\mathrm{d}(1/x)}} = \lim_{x\to 0^+}\mathrm{e}^{\frac{1/x}{-1/x^2}} = \lim_{x\to 0^+}\mathrm{e}^{-x} = \mathrm{e}^{-0} = \mathrm{e}^0 = 1.$$

b. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$

First, I'll try plugging in the limiting value:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \left(1+\frac{1}{\infty}\right)^\infty = (1+0)^\infty = 1^\infty,$$

but this is indeterminate. So I'll turn everything into expressions with base e and see where that gets me:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\mathrm{e}^{x\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty}\mathrm{e}^{\frac{\ln(1+1/x)}{1/x}} = \mathrm{e}^{\frac{0}{0}}.$$

This is another case for L'Hôpital in the exponent:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\mathrm{e}^{\frac{\ln(1+1/x)}{1/x}} = \lim_{x\to\infty}\mathrm{e}^{\frac{\mathrm{d}\left(\ln(1+1/x)\right)}{\mathrm{d}(1/x)}} = \lim_{x\to\infty}\mathrm{e}^{\frac{\frac{\mathrm{d}(1+1/x)}{(1+1/x)}}{-\frac{\mathrm{d}x}{x^2}}}$$
$$= \lim_{x\to\infty}\mathrm{e}^{\frac{\frac{-1/x^2}{(1+1/x)}}{-1/x^2}} = \lim_{x\to\infty}\mathrm{e}^{\frac{1}{(1+1/x)}} = \mathrm{e}^{\frac{1}{(1+1/\infty)}} = \mathrm{e}^{\frac{1}{(1+0)}} = \mathrm{e}^1 = \mathrm{e}.$$

Due Problems

The following problems were due May 12 Thursday.

1 *Differentiate (find the differentials of) the following expressions. (Show at least one intermediate step for each.)*

a. $3x^2 e^{5x}$

$$\begin{aligned} d(3x^2 e^{5x}) &= e^{5x}\, d(3x^2) + 3x^2\, d(e^{5x}) = e^{5x}\big(3\, d(x^2)\big) + 3x^2\big(e^{5x}\, d(5x)\big) \\ &= 3e^{5x}(2x\, dx) + 3x^2 e^{5x}(5\, dx) = 6xe^{5x}\, dx + 15x^2 e^{5x}\, dx. \end{aligned}$$

b. 2^{3x+5y}

$$d(2^{3x+5y}) = 2^{3x+5y} \ln 2\, d(3x+5y) = 2^{3x+5y} \ln 2\,\big(d(3x) + d(5y)\big) = 2^{3x+5y} \ln 2\,(3\, dx + 5\, dy).$$

2 *Find the limit*

$$\lim_{x \to 1} x^{\frac{1}{1-x}}.$$

(Show at least what numerical calculation you make. If you use L′Hôpital′s Rule, then also show what differentials or derivatives you take.)

First,

$$\lim_{x \to 1} x^{\frac{1}{1-x}} = 1^{\frac{(1)}{1-(1)}} = 1^{\frac{1}{0}}$$

can't be done directly. Next,

$$\lim_{x \to 1} x^{\frac{1}{1-x}} = \lim_{x \to 1} e^{\frac{\ln x}{1-x}} = e^{\frac{\ln (1)}{1-(1)}} = e^{\frac{0}{0}}$$

is indeterminate. Using L'Hôpital's Rule,

$$\lim_{x \to 1} x^{\frac{1}{1-x}} = \lim_{x \to 1} e^{\frac{\ln x}{1-x}} = \lim_{x \to 1} e^{\frac{d(\ln x)}{d(1-x)}} = \lim_{x \to 1} e^{\frac{dx/x}{-dx}} = \lim_{x \to 1} e^{-\frac{1}{x}} = e^{-\frac{1}{(1)}} = e^{-1} = \frac{1}{e}.$$