Practice Problems

These problems are not to be handed in, but try them first; do as many of them as you need until they're easy, or make up more along the same lines if you need more practice.

1 Differentiate the following expressions.

a. 3^{x^2+1}

b. $\ln(xy+2)$

c. $(x+2)^{3x}$

d. $\log_5(p+q)$

e. $5x^2\mathrm{e}^{2x-7}$

f. $\log_x(x+1)$

2 Find the following limits.

a. $\lim_{x \to 0^+} (x^x)$

b. $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

Due Problems

These problems are due November 6 Tuesday.

1 Differentiate (find the differentials of) the following expressions. (Show at least one intermediate step for each.)

a. $5x\mathrm{e}^{3x^2}$

b. $\log_2(4x-9y)$

2 Find the limit

$$\lim_{x \to 1} x^{\frac{1}{1-x}}.$$

Show at least what numerical calculation (or calculation with infinities) you make. If you use L'Hôpital's Rule, then also show the relevant differentials (or derivatives).

Answers to Practice Problems

Here are the answers to the Practice Problems from the beginning of the assignment.

1

a. I can use the rule for a constant base if I remember it:

$$\mathrm{d}\left(3^{x^2+1}\right) = 3^{x^2+1} \ln 3 \, \mathrm{d}(x^2+1) = 3^{x^2+1} \ln 3 \, (2x \, \mathrm{d}x) = 2x 3^{x^2+1} \ln 3 \, \mathrm{d}x.$$

Or I can turn everything into expressions with base e, using a simpler differentiation rule but more algebra:

$$\begin{aligned}\mathrm{d}\left(3^{x^2+1}\right) &= \mathrm{d}\left(\mathrm{e}^{(x^2+1)\ln 3}\right) = \mathrm{e}^{(x^2+1)\ln 3} \, \mathrm{d}\big((x^2+1)\ln 3\big) \\ &= 3^{x^2+1} \ln 3 \, \mathrm{d}(x^2+1) = 3^{x^2+1} \ln 3 \, (2x \, \mathrm{d}x) = 2x 3^{x^2+1} \ln 3 \, \mathrm{d}x.\end{aligned}$$

b. Since the base is already e, there's really only one way to do this:

$$\mathrm{d}(\ln(xy+2)) = \frac{\mathrm{d}(xy+2)}{xy+2} = \frac{y \, \mathrm{d}x + x \, \mathrm{d}y}{xy+2}.$$

c. Since the base and exponent are both variable, I'll turn everything into expressions with base e:

$$\begin{aligned}\mathrm{d}\left((x+2)^{3x}\right) &= \mathrm{d}\left(\mathrm{e}^{3x\ln(x+2)}\right) = \mathrm{e}^{3x\ln(x+2)} \, \mathrm{d}(3x\ln(x+2)) \\ &= (x+2)^{3x}\Big(3\ln(x+2)\,\mathrm{d}x + 3x\,\mathrm{d}\big(\ln(x+2)\big)\Big) \\ &= (x+2)^{3x}\left(3\ln(x+2)\,\mathrm{d}x + 3x\frac{\mathrm{d}(x+2)}{x+2}\right) \\ &= (x+2)^{3x}\left(3\ln(x+2)\,\mathrm{d}x + 3x\frac{\mathrm{d}x}{x+2}\right).\end{aligned}$$

(It would also be possible to do this with one complicated rule.)

d. As in part (a), I can use the rule for a constant base if I remember it:

$$\mathrm{d}(\log_5(p+q)) = \frac{\mathrm{d}(p+q)}{(p+q)\ln 5} = \frac{\mathrm{d}p + \mathrm{d}q}{(p+q)\ln 5}.$$

Or I can turn everything into expressions with base e, using a simpler differentiation rule but more algebra:

$$\mathrm{d}(\log_5(p+q)) = \mathrm{d}\left(\frac{\ln(p+q)}{\ln 5}\right) = \frac{\mathrm{d}(\ln(p+q))}{\ln 5} = \frac{\frac{\mathrm{d}(p+q)}{p+q}}{\ln 5} = \frac{\mathrm{d}p + \mathrm{d}q}{(p+q)\ln 5}.$$

e. I use the Product Rule first, then tackle the exponential part of the expression:

$$\begin{aligned}\mathrm{d}(5x^2\mathrm{e}^{2x-7}) &= \mathrm{e}^{2x-7}\,\mathrm{d}(5x^2) + 5x^2\,\mathrm{d}(\mathrm{e}^{2x-7}) = \mathrm{e}^{2x-7}(10x\,\mathrm{d}x) + 5x^2\mathrm{e}^{2x-7}\,\mathrm{d}(2x-7) \\ &= 10x\mathrm{e}^{2x-7}\,\mathrm{d}x + 5x^2\mathrm{e}^{2x-7}(2\,\mathrm{d}x) = 10x\mathrm{e}^{2x-7}\,\mathrm{d}x + 10x^2\mathrm{e}^{2x-7}\,\mathrm{d}x.\end{aligned}$$

f. As in part (c), I'll turn everything into expressions with base e:

$$\mathrm{d}\Big(\log_x(x+1)\Big) = \mathrm{d}\left(\frac{\ln(x+1)}{\ln x}\right) = \frac{\ln x\,\mathrm{d}(\ln(x+1)) - \ln(x+1)\,\mathrm{d}(\ln x)}{(\ln x)^2}$$
$$= \frac{\ln x\,\frac{\mathrm{d}(x+1)}{x+1} - \ln(x+1)\,\frac{\mathrm{d}x}{x}}{(\ln x)^2} = \frac{\ln x\,\frac{\mathrm{d}x}{x+1} - \ln(x+1)\,\frac{\mathrm{d}x}{x}}{(\ln x)^2}.$$

(It would also be possible to do this with one complicated rule.)

2

a. First, I'll try plugging in the limiting value:

$$\lim_{x\to 0^+}(x^x) = 0^0,$$

but this is indeterminate. So I'll turn everything into expressions with base e and see where that gets me:

$$\lim_{x\to 0^+}(x^x) = \lim_{x\to 0^+} \mathrm{e}^{x\ln x} = \lim_{x\to 0^+} \mathrm{e}^{\frac{\ln x}{1/x}} = \mathrm{e}^{\frac{-\infty}{\infty}}.$$

This is a case for L'Hôpital in the exponent:

$$\lim_{x\to 0^+}(x^x) = \lim_{x\to 0^+} \mathrm{e}^{\frac{\ln x}{1/x}} = \lim_{x\to 0^+} \mathrm{e}^{\frac{\mathrm{d}(\ln x)}{\mathrm{d}(1/x)}} = \lim_{x\to 0^+} \mathrm{e}^{\frac{1/x}{-1/x^2}} = \lim_{x\to 0^+} \mathrm{e}^{-x} = \mathrm{e}^{-0} = \mathrm{e}^0 = 1.$$

b. First, I'll try plugging in the limiting value:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \left(1+\frac{1}{\infty}\right)^\infty = (1+0)^\infty = 1^\infty,$$

but this is indeterminate. So I'll turn everything into expressions with base e and see where that gets me:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} \mathrm{e}^{x\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty} \mathrm{e}^{\frac{\ln(1+1/x)}{1/x}} = \mathrm{e}^{\frac{0}{0}}.$$

This is another case for L'Hôpital in the exponent:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} \mathrm{e}^{\frac{\ln(1+1/x)}{1/x}} = \lim_{x\to\infty} \mathrm{e}^{\frac{\mathrm{d}\left(\ln(1+1/x)\right)}{\mathrm{d}(1/x)}} = \lim_{x\to\infty} \mathrm{e}^{\frac{\frac{\mathrm{d}(1+1/x)}{(1+1/x)}}{-\frac{\mathrm{d}x}{x^2}}}$$
$$= \lim_{x\to\infty} \mathrm{e}^{\frac{\frac{-1/x^2}{(1+1/x)}}{-1/x^2}} = \lim_{x\to\infty} \mathrm{e}^{\frac{1}{(1+1/x)}} = \mathrm{e}^{\frac{1}{(1+1/\infty)}} = \mathrm{e}^{\frac{1}{(1+0)}} = \mathrm{e}^1 = \mathrm{e}.$$