

Practice Problems

These problems are not to be handed in, but try them first; do as many of them as you need until they're easy, or make up more along the same lines if you need more practice.

Approximate the following integrals using Riemann sums with 8 terms.

1 $\int_1^2 \frac{dt}{t}$

2 $\int_0^1 \frac{dx}{x^2 + 1}$

3 $\int_{-1}^1 e^{-z^2/2} dz$

4 $\int_0^1 \sqrt{1 - x^2} dx$

Due Problems

These problems are due December 4 Tuesday.

For each of the following integrals, approximate it using Riemann sums with 4 terms; use either a lower or upper Riemann sum. **Extra credit:** Do both, and state how precisely you have the true value of the integral.

1 $\int_0^4 \sqrt{x^2 + 3} dx$

2 $\int_1^2 \frac{x^2}{x + 5} dx$

Answers to Practice Problems

Here are the answers to the Practice Problems from the beginning of the assignment.

- 1** First, a technicality: I take the integrand dt/t , divide by dt to get $1/t$, and differentiate to get $-1/t^2 dt$. Since $-1/t^2$ is negative for t between 1 and 2, I know that $1/t$ is decreasing as t increases there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of t while the upper Riemann sum will come by using smaller values of t .

If I divide the interval from 1 to 2 into 8 pieces, then each piece has a width of $\frac{2-1}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $t = 1 + 0.125$ to $t = 2$, always rounding down; for the upper Riemann sum, I'll calculate heights from $t = 1$ to $t = 2 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

t	lower Riemann sum	upper Riemann sum
1	(not used)	1
1.125	0.888	0.889
1.25	0.8	0.8
1.375	0.727	0.728
1.5	0.666	0.667
1.625	0.615	0.616
1.75	0.571	0.572
1.875	0.533	0.534
2	0.5	(not used)
total	5.300	5.806

Therefore, the lower Riemann sum is $5.300 \cdot 0.125 \approx 0.662$ (rounding down), while the upper Riemann sum is $5.806 \cdot 0.125 \approx 0.726$ (rounding up). (You could get slightly different estimates, depending on how many digits you keep when rounding.)

Therefore, I conclude that

$$\int_1^2 \frac{dt}{t} \approx 0.7$$

to one decimal place. (As it happens, the correct value is $\ln 2 \approx 0.693$, which a scientific calculator can find quickly and efficiently using a variation of the method above.)

- 2** First, a technicality: I take the integrand $dx/(x^2 + 1)$, divide by dx to get $1/(x^2 + 1)$, and differentiate to get $-2x/(x^2 + 1)^2 dx$. Since $-2x/(x^2 + 1)^2$ is negative for x between 0 and 1, I know that $1/(x^2 + 1)$ is decreasing as x increases there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of x while the upper Riemann sum will come by using smaller values of x .

If I divide the interval from 0 to 1 into 8 pieces, then each piece has a width of $\frac{1-0}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $x = 0 + 0.125$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights from $x = 0$ to

$x = 1 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
0	(not used)	1
0.125	0.984	0.985
0.25	0.941	0.942
0.375	0.876	0.877
0.5	0.8	0.8
0.625	0.719	0.720
0.75	0.64	0.64
0.875	0.566	0.567
1	0.5	(not used)
total	6.026	6.531

Therefore, the lower Riemann sum is $6.026 \cdot 0.125 \approx 0.753$ (rounding down), while the upper Riemann sum is $6.531 \cdot 0.125 \approx 0.817$ (rounding up). (You could get slightly different estimates, depending on how many digits you keep when rounding.)

Therefore, I conclude that

$$\int_0^1 \frac{dx}{x^2 + 1} \approx 0.8$$

to one decimal place. (As it happens, the correct value is $\pi/4 \approx 0.785$, which you can calculate using trigonometry.)

- 3** First, a technicality: I take the integrand $e^{-z^2/2} dz$, divide by dz to get $e^{-z^2/2}$, and differentiate to get $-ze^{-z^2/2} dz$. Since $-ze^{-z^2/2}$ is positive for x between -1 and 0 , I know that $e^{-z^2/2}$ is increasing as z increases there; since $-ze^{-z^2/2}$ is negative for x between 0 and 1 , I know that $e^{-z^2/2}$ is decreasing as z increases there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using first smaller values of z and then larger values, while the upper Riemann sum will come by using first larger values of z and then smaller values.

If I divide the interval from -1 to 1 into 8 pieces, then each piece has a width of $\frac{1-(-1)}{8} = 0.25$. For the lower Riemann sum, I'll calculate heights from $x = -1$ to $x = 0 - 0.25$ and then from $x = 0 + 0.25$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights from $x = -1 + 0.25$ to $x = 0$ and then from $x = 0$ (again!) to $x = 1 - 0.25$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.25. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
-1	0.606	(not used)
-0.75	0.754	0.755
-0.5	0.882	0.883
-0.25	0.969	0.970
0	(not used)	1 (used twice)
0.25	0.969	0.970

0.5	0.882	0.883
0.75	0.754	0.755
1	0.606	(not used)
total	6.422	7.216

Therefore, the lower Riemann sum is $6.422 \cdot 0.25 \approx 1.605$ (rounding down), while the upper Riemann sum is $7.216 \cdot 0.25 = 1.804$. (You could get slightly different estimates, depending on how many digits you keep when rounding.)

Therefore, I conclude that

$$\int_{-1}^1 e^{-z^2/2} dz \approx 1.7,$$

give or take one decimal place. (As it happens, the correct value is $\sqrt{2\pi} \operatorname{erf}(\sqrt{2}/2) \approx 1.711$, which you can calculate using a statistics calculator.)

- 4 First, a technicality: I take the integrand $\sqrt{1-x^2} dx$, divide by dx to get $\sqrt{1-x^2}$, and differentiate to get $-x\sqrt{1-x^2}/(1-x^2) dx$. Since $-x\sqrt{1-x^2}/(1-x^2)$ is negative for x between 0 and 1, I know that $\sqrt{1-x^2}$ is decreasing as x increases there. (Really, you could see this by drawing a quick graph.) Thus, the lower Riemann sum will come by using larger values of x while the upper Riemann sum will come by using smaller values of x .

If I divide the interval from 0 to 1 into 8 pieces, then each piece has a width of $\frac{1-0}{8} = 0.125$. For the lower Riemann sum, I'll calculate heights from $x = 0 + 0.125$ to $x = 1$, always rounding down; for the upper Riemann sum, I'll calculate heights from $x = 0$ to $x = 1 - 0.125$, always rounding up. Either way, I'll add up all of these heights and multiply the total by 0.125. The calculations are in the following table:

x	lower Riemann sum	upper Riemann sum
0	(not used)	1
0.125	0.992	0.993
0.25	0.968	0.969
0.375	0.927	0.928
0.5	0.866	0.867
0.625	0.780	0.781
0.75	0.661	0.662
0.875	0.484	0.485
1	0	(not used)
total	5.678	6.685

Therefore, the lower Riemann sum is $5.678 \cdot 0.125 \approx 0.709$ (rounding down), while the upper Riemann sum is $6.685 \cdot 0.125 \approx 0.836$ (rounding up). (You could get slightly different estimates, depending on how many digits you keep when rounding.)

Therefore, I conclude that

$$\int_0^1 \sqrt{1-x^2} dx \approx 0.8$$

give or take one decimal place. (As it happens, the correct value is $\pi/4 \approx 0.785$, which you can calculate using trigonometry.)