Practice Problems

These problems are not to be handed in, but try them first; do as many of them as you need until they're easy, or make up more along the same lines if you need more practice.

1 *Find the second derivative of y with respect to x:*

a. $y = 5x^3 - 4x^2 + 3x$

Following problem (3.a) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^2 - 8x + 3.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}(15x^2 - 8x + 3) = 30x\,\mathrm{d}x - 8\,\mathrm{d}x,$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = 30x - 8.$$

b. $y = \dfrac{12}{x+5} - 10$

Following problem (3.b) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{12}{(x+5)^2} = -12(x+5)^{-2}.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}\Big(-12(x+5)^{-2}\Big) = -12\Big(-2(x+5)^{-3}\,\mathrm{d}(x+5)\Big) = \frac{24\,\mathrm{d}x}{(x+5)^3},$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = \frac{24}{(x+5)^3}.$$

c. $x^2 + y^2 = 1$

Following problem (3.c) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}\left(-\frac{x}{y}\right) = -\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{y^2} = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{y^2},$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = \frac{x\,\mathrm{d}y/\mathrm{d}x - y}{y^2} = \frac{x(-x/y) - y}{y^2} = \frac{-x^2 - y^2}{y^3} = -\frac{x^2 + y^2}{y^3}.$$

d. $(x+y)^2 = 1$

Following problem (3.d) in Homework 4,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1.$$

Therefore,

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{d}(-1) = 0,$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} = 0.$$

2 *Find formulas for f' and f'':*

a. $f(x) = 3x^3$

I differentiate the formula for $f(x)$ and divide by $\mathrm{d}x$:

$$\begin{aligned} f(x) &= 3x^3; \\ \mathrm{d}f(x) &= \mathrm{d}(3x^3) = 9x^2\,\mathrm{d}x; \\ f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= 9x^2. \end{aligned}$$

To find f'', I do this again:

$$\begin{aligned} f'(x) &= 9x^2; \\ \mathrm{d}f'(x) &= \mathrm{d}(9x^2) = 18x\,\mathrm{d}x; \\ f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= 18x. \end{aligned}$$

b. $f(t) = \sqrt{t-5}$

I differentiate the formula for $f(t)$ and divide by $\mathrm{d}t$:

$$\begin{aligned} f(t) &= \sqrt{t-5}; \\ \mathrm{d}f(t) &= \mathrm{d}\big(\sqrt{t-5}\big) = \frac{\sqrt{t-5}\,\mathrm{d}(t-5)}{2(t-5)} = \frac{\sqrt{t-5}\,\mathrm{d}t}{2t-10}; \\ f'(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} &= \frac{\sqrt{t-5}}{2t-10}. \end{aligned}$$

To find f'', I do this again:

$$\begin{aligned} f'(t) &= \frac{\sqrt{t-5}}{2t-10}; \\ \mathrm{d}f'(t) &= \mathrm{d}\left(\frac{\sqrt{t-5}}{2t-10}\right) = \frac{(2t-10)\,\mathrm{d}\big(\sqrt{t-5}\big) - \sqrt{t-5}\,\mathrm{d}(2t-10)}{(2t-10)^2} \\ &= \frac{\sqrt{t-5}\,\mathrm{d}t - 2\sqrt{t-5}\,\mathrm{d}t}{(2t-10)^2} = -\frac{\sqrt{t-5}\,\mathrm{d}t}{(2t-10)^2}; \\ f''(t) = \frac{\mathrm{d}f'(t)}{\mathrm{d}t} &= -\frac{\sqrt{t-5}}{(2t-10)^2}. \end{aligned}$$

c. $f(x) = \dfrac{x^2}{x-1}$

I differentiate the formula for $f(x)$ and divide by $\mathrm{d}x$:

$$\begin{aligned} f(x) &= \frac{x^2}{x-1}; \\ \mathrm{d}f(x) &= \mathrm{d}\left(\frac{x^2}{x-1}\right) = \frac{(x-1)\,\mathrm{d}(x^2) - x^2\,\mathrm{d}(x-1)}{(x-1)^2} = \frac{2x(x-1)\,\mathrm{d}x - x^2\,\mathrm{d}x}{(x-1)^2}; \\ f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}. \end{aligned}$$

To find f'', I do this again:

$$
\begin{aligned}
f'(x) &= \frac{x^2 - 2x}{(x-1)^2} = (x^2 - 2x)(x-1)^{-2}; \\
\mathrm{d}f'(x) &= \mathrm{d}\big((x^2 - 2x)(x-1)^{-2}\big) \\
&= (x-1)^{-2}\,\mathrm{d}(x^2 - 2x) + (x^2 - 2x)\,\mathrm{d}\big((x-1)^{-2}\big) \\
&= (x-1)^{-2}(2x\,\mathrm{d}x - 2\,\mathrm{d}x) - 2(x^2 - 2x)(x-1)^{-3}\,\mathrm{d}x \\
&= \frac{2x\,\mathrm{d}x - 2\,\mathrm{d}x}{(x-1)^2} - \frac{2(x^2 - 2x)\,\mathrm{d}x}{(x-1)^3}; \\
f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= \frac{2x - 2}{(x-1)^2} - \frac{2(x^2 - 2x)}{(x-1)^3}.
\end{aligned}
$$

(This could be simplified further, if you wish.)

d. $f(p) = 5 - p^2$

I differentiate the formula for $f(p)$ and divide by $\mathrm{d}p$:

$$
\begin{aligned}
f(p) &= 5 - p^2; \\
\mathrm{d}f(p) &= \mathrm{d}(5 - p^2) = -2p\,\mathrm{d}p; \\
f'(p) = \frac{\mathrm{d}f(p)}{\mathrm{d}p} &= -2p.
\end{aligned}
$$

To find f'', I do this again:

$$
\begin{aligned}
f'(p) &= -2p; \\
\mathrm{d}f'(p) &= \mathrm{d}(-2p) = -2\,\mathrm{d}p; \\
f''(p) = \frac{\mathrm{d}f'(p)}{\mathrm{d}p} &= -2.
\end{aligned}
$$

Due Problems

These problems are due October 18 Thursday.

1 *Given that*

$$y = 7x^4 - 12x^3$$

always, find the first and second derivatives of y with respect to x. (Show at least one intermediate step for each.)

To find the first derivative, I differentiate the equation and divide by $\mathrm{d}x$:

$$
\begin{aligned}
\mathrm{d}y &= \mathrm{d}(7x^4 - 12x^3) \\
&= 7(4x^3\,\mathrm{d}x) - 12(3x^2\,\mathrm{d}x); \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 28x^3 - 36x^2.
\end{aligned}
$$

To find the second derivative, I repeat the process:

$$
\begin{aligned}
\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= \mathrm{d}(28x^3 - 36x^2) \\
&= 28(3x^2\,\mathrm{d}x) - 36(2x\,\mathrm{d}x); \\
\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} &= 84x^2 - 72x.
\end{aligned}
$$

2 *Given that*

$$f(t) = \sqrt[3]{t} - \frac{5}{t}$$

always, find the first and second derivatives of the function f. (Show at least one intermediate step for each.)

To find the first derivative, I differentiate the formula and divide by $\mathrm{d}t$:

$$\begin{aligned} \mathrm{d}f(t) &= \mathrm{d}\Big(\sqrt[3]{t} - \frac{5}{t}\Big) \\ &= \frac{\sqrt[3]{t}\,\mathrm{d}t}{3t} - \Big(-\frac{5\,\mathrm{d}t}{t^2}\Big) \\ f'(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} &= \frac{\sqrt[3]{t}}{3t} + \frac{5}{t^2}. \end{aligned}$$

To find the second derivative, I repeat the process:

$$\begin{aligned} \mathrm{d}f'(t) &= \mathrm{d}\Big(\frac{\sqrt[3]{t}}{3t} + \frac{5}{t^2}\Big) = \mathrm{d}\Big(\frac{1}{3}t^{-2/3} + 5t^{-2}\Big) \\ &= \frac{1}{3}\Big(-\frac{2}{3}t^{-5/3}\,\mathrm{d}t\Big) + 5(-2t^{-3}\,\mathrm{d}t); \\ f''(t) = \frac{\mathrm{d}f'(t)}{\mathrm{d}t} &= -\frac{2}{9}t^{-5/3} - 10t^{-3} = -\frac{2\sqrt[3]{t}}{9t^2} - \frac{10}{t^3}. \end{aligned}$$

3 **Extra credit:** *In Physics classes, one learns the formula*

$$y = a + bt - \frac{1}{2}gt^2$$

for the height y of a projectile that is released at a height a with upward velocity b, where t is the amount of time after the projectile is released and g is the local downward acceleration of gravity. (Note that a, b, and g are constant in this situation.) Verify that this formula is correct (showing in each case what calculation you must make to verify this), as follows:

a. *At the time at which the projectile is released, check that the height really is a. (Hint: $t = 0$ when the projectile is released.)*

When $t = 0$,

$$y = a + b(0) - \frac{1}{2}g(0)^2 = a,$$

as claimed.

b. *At the time at which the projectile is released, check that the upward velocity really is b. (Hint: Upward velocity is the derivative of height with respect to time.)*

At all times,

$$\begin{aligned} \mathrm{d}y &= \mathrm{d}\Big(a + bt - \frac{1}{2}gt^2\Big) \\ &= 0 + b\,\mathrm{d}t - \frac{1}{2}g(2t\,\mathrm{d}t); \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= b - gt. \end{aligned}$$

At $t = 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = b - g(0) = b,$$

as claimed.

c. *At all times, check that the downward acceleration of the particle really is g. (Hint: Upward acceleration is the second derivative of height with respect to time, and downward acceleration is simply the opposite of upward acceleration.)*

At all times,

$$\begin{aligned} \mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) &= \mathrm{d}(b - gt) \\ &= 0 - g\,\mathrm{d}t; \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}t)}{\mathrm{d}t} &= -g. \end{aligned}$$

Therefore,

$$-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 y = -(-g) = g,$$

as claimed.