Practice Problems

These problems are not to be handed in, but try them first; also try the even problems if you need more practice.

- From §5-5 (pages 330&331): 1–9 odd, 11, 15, 17–31 odd;
- From §5-6 (pages 340&343): 11, 13, 19–29 odd, 33, 35, 37.

The answers to these should be in the back of your textbook.

Due Problems

These problems are due October 30 Tuesday.

In each problem (or part thereof), show at least *what algebraic equation or equations you solve, as well as your final answer in words (with correct units).*

1 *Suppose that research for a small automobile company suggests that the annual revenue from selling x cars per year will be $40\,000\,x - 50\,x^2$ dollars, while the annual cost of producing x cars per year will be $1000\,000 + 10\,000\,x$ dollars.*

a. *If the marketing department tries to maximise revenue, what goal will they set as the number of cars to sell in a year?*

$$\mathrm{d}R = \mathrm{d}(40000x - 50x^2) = 40000\,\mathrm{d}x - 100x\,\mathrm{d}x = (40000 - 100x)\,\mathrm{d}x.$$

As long as x can vary smoothly, this is never undefined, and it's zero only if $40000 - 100x = 0$, which gives $x = 400$.

This is probably the answer, but to be sure, I can check the extreme cases or limits. If $x = 0$, then $R = 0$; as $x \to \infty$, then

$$R = x^2(40000/x - 50) \to \infty^2(0 - 50) = -\infty.$$

But if $x = 400$, then $R = 8\,000\,000$, which is largest. So indeed, they will try to sell 400 **cars per year**.

b. *How many cars should actually be manufactured and sold in a year in order to maximise profit for the company?*

$$P = R - C = 30000x - 50x^2 - 1000000;$$
$$\mathrm{d}P = (30000 - 100x)\,\mathrm{d}x.$$

As long as x can vary smoothly, this is never undefined, and it's zero only if $30000 - 100x = 0$, which gives $x = 300$.

This is probably the answer, but to be sure, I can check the extreme cases or limits. If $x = 0$, then $P = -1000000$; as $x \to \infty$, then

$$P = x^2\left(30000/x - 50 - \frac{1000000}{x^2}\right) \to \infty^2(0 - 50 - 0) = -\infty.$$

But if $x = 300$, then $P = 4\,500\,000$, which is largest. So indeed, they will try to make and sell 300 **cars per year**.

2 *In the countryside between two polluted metropolises (LA and SD), the pollution is worse nearer the city and better farther out in the country. A simple model of pollution says that the pollution from LA is* $800/x^2$ ppm *at a point* x *miles from LA, while the pollution from SD is* $1000/y^2$ ppm *at a point* y *miles from SD. The distance between LA and SD is* 100 *miles.*

a. *Using this model, what point between LA and SD has the lowest total pollution? (Give the distance of this point from LA or SD, and say which distance it is.)*

Between LA and SD, we have

$$x + y = 100,$$

so

$$\mathrm{d}x + \mathrm{d}y = 0.$$

The total pollution is

$$P = \frac{800}{x^2} + \frac{1000}{y^2} = 800x^{-2} + 1000y^{-2},$$

so

$$\mathrm{d}P = -1600x^{-3}\,\mathrm{d}x - 2000y^{-3}\,\mathrm{d}y = -\frac{1600\,\mathrm{d}x}{x^3} - \frac{2000\,\mathrm{d}y}{y^3}.$$

This differential $\mathrm{d}P$ is undefined only in the extreme cases where $x = 0$ or $y = 0$. The formula for P doesn't apply there directly, but we can say that

$$\lim_{x\to0^+} P = \lim_{x\to0^+} \frac{800}{x^2} + \frac{1000}{(100-x)^2} = \infty$$

and that

$$\lim_{y\to0^+} P = \lim_{y\to0^+} \frac{800}{(100-y)^2} + \frac{1000}{y^2} = \infty.$$

Therefore, there is no maximum value of P; realistically, the formula for P can't be correct when we get to close to LA or to SD (presumably within the region where the pollution is being produced).

The only possibility left for the minimum of P is when $\mathrm{d}P = 0$. From

$$\mathrm{d}x + \mathrm{d}y = 0,$$

I get

$$\mathrm{d}y = -\mathrm{d}x,$$

so

$$0 = \mathrm{d}P = -\frac{1600\,\mathrm{d}x}{x^3} - \frac{2000\,\mathrm{d}y}{y^3} = \frac{2000\,\mathrm{d}x}{y^3} - \frac{1600\,\mathrm{d}x}{x^3}.$$

Since $\mathrm{d}x \neq 0$ between LA and SD, I get

$$2000x^3 = 1600y^3,$$

so

$$y = \sqrt[3]{\frac{5}{4}}x.$$

Then

$$x + \sqrt[3]{\frac{5}{4}}x = 100,$$

so

$$x = \frac{100}{1 + \sqrt[3]{\frac{5}{4}}} \approx 48,$$

and

$$y = 100 - \frac{100}{1 + \sqrt[3]{\frac{5}{4}}} = \frac{100}{1 + \sqrt[3]{\frac{4}{5}}} \approx 52.$$

There are no other possible extrema, so the minimum must be here.

Therefore, the pollution is minimal **about** 48 **miles from LA**, or **about** 52 **miles from SD**.

b. *How much pollution is at this point?*

When $x \approx 48$ and $y \approx 52$,

$$P \approx \frac{800}{48^2} + \frac{1000}{52^2} \approx 0.717,$$

so the minimal pollution is **about** 0.717 **parts per million**, or **about** 717 **parts per thousand**. (Unrounded, the minimal value of P is

$$\frac{2}{25}\left(1 + \sqrt[3]{\frac{5}{4}}\right)^2 + \frac{1}{10}\left(1 + \sqrt[3]{\frac{4}{5}}\right)^2,$$

but you probably wouldn't want to bother with that.)