This exam runs from 12:00 to 1:25 PM; if I've written it well, then you shouldn't need the whole time. You may use one sheet of notes that you've written yourself, but not your textbook, handouts, or anything else not written by you, and you may not communicate with anybody but me. You can come up and talk to me if you have questions, especially about the instructions. Also, you may use a calculator if you wish, although you shouldn't need one.

Take your time, and check over your answers. Read the instructions carefully, and be sure to show everything that they ask. You can always show *more* work if you like; for instance, you may draw a graph if you find it helpful, even when a problem does not require you to draw a graph. If you're unsure of your answer, then explain what you're unsure about, and show your work so that you can get as much partial credit as possible.

Some of these problems consists of several related parts. You don't have to do those parts in order one after another; depending on how you think, it may be easier to do them in a different order or even to go back and forth. But make sure that the final answer to each part is clear.

Don't forget to put your name on the exam!!!

1 *In each part, you are given an equation relating x and y. Find the derivative of y with respect to x; that is, find an expression for $\mathrm{d}y/\mathrm{d}x$. (Show at least one intermediate step for each.)*

a $y = 3(x-4)^2$

Differentiate to find $\mathrm{d}y$, then divide by $\mathrm{d}x$:

$$\begin{aligned} y &= 3(x-4)^2; \\ \mathrm{d}y &= \mathrm{d}\big(3(x-4)^2\big) \\ &= 3\,\mathrm{d}\big((x-4)^2\big) \\ &= 3\big(2(x-4)^{2-1}\,\mathrm{d}(x-4)\big) \\ &= 6(x-4)\,\mathrm{d}x; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 6(x-4). \end{aligned}$$

b $y = 3\sqrt{x-4}$

Again, differentiate to find $\mathrm{d}y$, then divide by $\mathrm{d}x$:

$$\begin{aligned} y &= 3\sqrt{x-4}; \\ \mathrm{d}y &= \mathrm{d}\big(3\sqrt{x-4}\big) \\ &= 3\,\mathrm{d}\big(\sqrt{x-4}\big) \\ &= 3\frac{\sqrt{x-4}\,\mathrm{d}(x-4)}{2(x-4)} \\ &= \frac{3\sqrt{x-4}\,\mathrm{d}x}{2(x-4)}; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3\sqrt{x-4}}{2(x-4)}. \end{aligned}$$

c $5x + 6y = x^5 + y^2$

This time, differentiate and solve for $\mathrm{d}y$, then divide by $\mathrm{d}x$:

$$\begin{aligned}
5x + 6y &= x^5 + y^2; \\
\mathrm{d}(5x + 6y) &= \mathrm{d}(x^5 + y^2); \\
\mathrm{d}(5x) + \mathrm{d}(6y) &= \mathrm{d}(x^5) + \mathrm{d}(y^2); \\
5\,\mathrm{d}x + 6\,\mathrm{d}y &= 5x^{5-1}\,\mathrm{d}x + 2y^{2-1}\,\mathrm{d}y; \\
6\,\mathrm{d}y - 2y\,\mathrm{d}y &= 5x^4\,\mathrm{d}x - 5\,\mathrm{d}x; \\
(6 - 2y)\,\mathrm{d}y &= (5x^4 - 5)\,\mathrm{d}x; \\
\mathrm{d}y &= \frac{(5x^4 - 5)\,\mathrm{d}x}{6 - 2y}; \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{5x^4 - 5}{6 - 2y}.
\end{aligned}$$

2 *Find the first and second derivatives of each of the following functions; that is, find formulas for both f' and f''. (Show at least one intermediate step for each.)*

a $f(x) = 4x^3 + 2x^2$

Differentiate to find $\mathrm{d}f(x)$, then divide by $\mathrm{d}x$ to find $f'(x)$:

$$\begin{aligned}
f(x) &= 4x^3 + 2x^2; \\
\mathrm{d}f(x) &= \mathrm{d}(4x^3 + 2x^2) \\
&= \mathrm{d}(4x^3) + \mathrm{d}(2x^2) \\
&= 4\,\mathrm{d}(x^3) + 2\,\mathrm{d}(x^2) \\
&= 4(3x^{3-1}\,\mathrm{d}x) + 2(2x^{2-1}\,\mathrm{d}x) \\
&= 12x^2\,\mathrm{d}x + 4x\,\mathrm{d}x; \\
f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= 12x^2 + 4x.
\end{aligned}$$

Now differentiate again to find $\mathrm{d}f'(x)$, then divide by $\mathrm{d}x$ again to find $f''(x)$:

$$\begin{aligned}
f'(x) &= 12x^2 + 4x; \\
\mathrm{d}f'(x) &= \mathrm{d}(12x^2 + 4x) \\
&= \mathrm{d}(12x^2) + \mathrm{d}(4x) \\
&= 12\,\mathrm{d}(x^2) + 4\,\mathrm{d}x \\
&= 12(2x^{2-1}\,\mathrm{d}x) + 4\,\mathrm{d}x \\
&= 24x\,\mathrm{d}x + 4\,\mathrm{d}x \\
f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= 24x + 4.
\end{aligned}$$

b $f(x) = \dfrac{x+4}{x-1}$

Again, differentiate to find $\mathrm{d}f(x)$, then divide by $\mathrm{d}x$ to find $f'(x)$:

$$\begin{aligned}
f(x) &= \frac{x+4}{x-1};\\
\mathrm{d}f(x) &= \mathrm{d}\left(\frac{x+4}{x-1}\right)\\
&= \frac{(x-1)\,\mathrm{d}(x+4) - (x+4)\,\mathrm{d}(x-1)}{(x-1)^2}\\
&= \frac{(x-1)\,\mathrm{d}x - (x+4)\,\mathrm{d}x}{(x-1)^2} = -\frac{5\,\mathrm{d}x}{(x-1)^2};\\
f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= -\frac{5}{(x-1)^2}.
\end{aligned}$$

Now differentiate again to find $\mathrm{d}f'(x)$, then divide by $\mathrm{d}x$ again to find $f''(x)$:

$$\begin{aligned}
f'(x) &= -\frac{5}{(x-1)^2} = -5(x-1)^{-2};\\
\mathrm{d}f'(x) &= \mathrm{d}\big(-5(x-1)^{-2}\big)\\
&= -5\,\mathrm{d}\big((x-1)^{-2}\big)\\
&= -5\big(-2(x-1)^{-2-1}\,\mathrm{d}(x-1)\big)\\
&= 10(x-1)^{-3}\,\mathrm{d}x;\\
f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} &= 10(x-1)^{-3} = \frac{10}{(x-1)^3}.
\end{aligned}$$

3 *Evaluate the following limits. (Show at least one intermediate step for each.)*

a $\lim\limits_{x\to 4}\left(\dfrac{x^2-16}{x-4}\right)$

First, try just plugging 4 in for x:

$$\lim_{x\to 4}\left(\frac{x^2-16}{x-4}\right) = \frac{(4)^2-16}{4-4} = \frac{0}{0};$$

but this is undefined. However, it's an indeterminate form, to which L'Hôpital's Rule applies, so use that:

$$\lim_{x\to 4}\left(\frac{x^2-16}{x-4}\right) = \lim_{x\to 4}\left(\frac{\mathrm{d}(x^2-16)}{\mathrm{d}(x-4)}\right) = \lim_{x\to 4}\left(\frac{2x\,\mathrm{d}x}{\mathrm{d}x}\right) = \lim_{x\to 4}(2x) = 2(4) = 8.$$

b $\lim\limits_{y\to-\infty}(y^3+5)$

First, try plugging $-\infty$ in for y and use the rules for calculating with ∞:

$$\lim_{y\to-\infty}(y^3+5) = (-\infty)^3 + 5 = -\infty^3 + 5 = -\infty + 5 = -\infty.$$

c $\lim_{x\to 2^+}\left(\frac{x^2-2}{x-2}\right)$

First, try just plugging 2 in for x:

$$\lim_{x\to 2^+}\left(\frac{x^2-2}{x-2}\right) = \frac{(2)^2-2}{2-2} = \frac{2}{0};$$

but this is undefined. However, we can divide by zero when calculating a limit if we pay attention to the signs: 2 is positive, and $x-2$ is positive when $x>2$ (which is always the case as $x\to 2^+$). Since positive divided by positive is positive, the limit is

$$\infty.$$

d **Extra credit:** $\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right)$*, where* f *is a smooth function and* c *is a constant*

First, try just plugging 0 in for h:

$$\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right) = \frac{f(c+0)-f(c)}{0} = \frac{f(c)-f(c)}{0} = \frac{0}{0};$$

but this is undefined. However, it's a form to which L'Hôpital's Rule applies, so use that, remembering that $\mathrm{d}f(x) = f'(x)\,\mathrm{d}x$:

$$\begin{aligned}\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right) &= \lim_{h\to 0}\left(\frac{\mathrm{d}(f(c+h)-f(c))}{\mathrm{d}h}\right) = \lim_{h\to 0}\left(\frac{\mathrm{d}f(c+h)-\mathrm{d}f(c)}{\mathrm{d}h}\right)\\ &= \lim_{h\to 0}\left(\frac{f'(c+h)\,\mathrm{d}(c+h)-f'(c)\,\mathrm{d}c}{\mathrm{d}h}\right) = \lim_{h\to 0}\left(\frac{f'(c+h)\,\mathrm{d}h-f'(c)(0)}{\mathrm{d}h}\right)\\ &= \lim_{h\to 0} f'(c+h) = f'(c+0) = f'(c).\end{aligned}$$

(This is the key insight that Augustin Cauchy had in 1821 to realise that one could *define* the derivative of a function as a limit, which is the opposite approach from what we are doing.)

4 *For each of the following functions, find its maximum and minimum value, if they exist. (Show at least what numerical calculation you make for each, as well as at least one other intermediate step.)*

a $f(x) = x^4 + 20x^3 + 100x^2$

First I find the derivative f':

$$\begin{aligned}f(x) &= x^4 + 20x^3 + 100x^2;\\ \mathrm{d}f(x) &= \mathrm{d}(x^4 + 20x^3 + 100x^2) = 4x^3\,\mathrm{d}x + 20(3x^2\,\mathrm{d}x) + 100(2x\,\mathrm{d}x);\\ f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= 4x^3 + 60x^2 + 200x.\end{aligned}$$

This is always defined, so I see when this is zero:

$$\begin{aligned}0 &= 4x^3 + 60x^2 + 200x = 4x(x^2+15x+50) = 4x(x+5)(x+10);\\ x &= 0 \text{ or } x+5=0 \text{ or } x+10=0;\\ x &= 0 \text{ or } x=-5 \text{ or } x=-10.\end{aligned}$$

So I check these, as well as the limits at $\pm\infty$:

- $f(0) = (0)^4 + 20(0)^3 + 100(0)^2 = 0;$
- $f(-5) = (-5)^4 + 20(-5)^3 + 100(-5)^2 = 625;$
- $f(-10) = (-10)^4 + 20(-10)^3 + 100(-10)^2 = 0;$
- $\lim_{x\to\infty} f(x) = (\infty)^4 + 20(\infty)^3 + 100(\infty)^2 = \infty;$
- $\lim_{x\to-\infty} f(x) = (-\infty)^4 + 20(-\infty)^3 + 100(-\infty)^2 = \infty.$

Of these results, 0 is the smallest and ∞ is the largest. Therefore, **the minimum value is** 0, and there is **no maximum value**.

b $f(x) = \sqrt{400 - x^2}$

First I find the derivative f':

$$f(x) = \sqrt{400 - x^2};$$
$$\mathrm{d}f(x) = \mathrm{d}\left(\sqrt{400 - x^2}\right) = \frac{\sqrt{400 - x^2}\,\mathrm{d}(400 - x^2)}{2(400 - x^2)} = \frac{\sqrt{400 - x^2}(-2x\,\mathrm{d}x)}{2(400 - x^2)};$$
$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\frac{x\sqrt{400 - x^2}}{400 - x^2}.$$

This is sometimes undefined:

$$400 - x^2 = 0;$$
$$x^2 = 400;$$
$$x = \pm 20.$$

Besides this, it is sometimes zero:

$$-x\sqrt{400 - x^2} = 0;$$
$$-x = 0;$$
$$x = 0.$$

Now, $f(x)$ itself is only defined sometimes:

$$400 - x^2 \geq 0;$$
$$x^2 \leq 400;$$
$$-20 \leq x \leq 20.$$

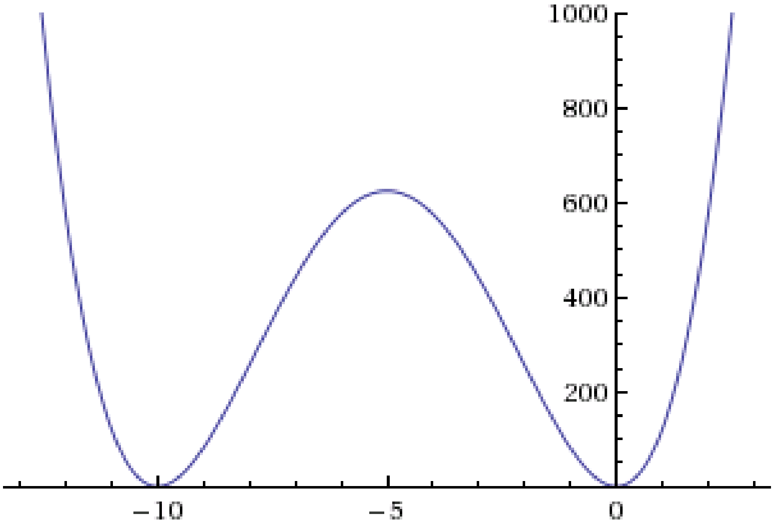
So I have only three values to check:

- $f(20) = \sqrt{400 - (20)^2} = 0;$
- $f(-20) = \sqrt{400 - (-20)^2} = 0;$
- $f(0) = \sqrt{400 - (0)^2} = 20.$

Of these results, 0 is the smallest and 20 is the largest. Therefore, **the minimum value is** 0, and **the maximum value is** 20.

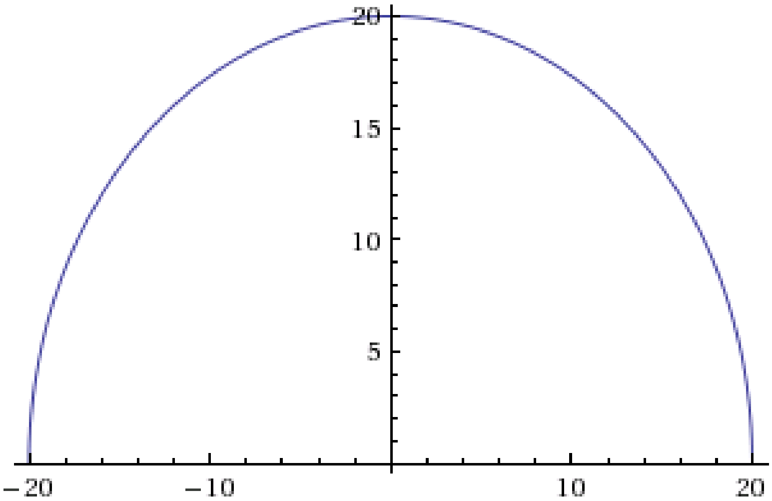
c *Sketch a graph of* one *of these functions, making sure that every local extremum appears on the graph. (Be sure to label the scale on your graph.)*

To graph $f(x) = x^4 + 20x^3 + 100x^2$, I need a window in which $-10 \leq x \leq 0$ (or a bit wider) and in which $0 \leq f(x) \leq 625$ (or a bit taller). I actually used $-13 \leq x \leq 3$ and $0 \leq f(x) \leq 1000$:



(This graph was produced to my instructions by Wolfram Alpha. There really should be arrows at the ends of the curve.)

To graph $f(x) = \sqrt{400 - x^2}$, I need a window in which $-20 \le x \le 20$ (or a bit wider) and in which $0 \le f(x) \le 20$ (or a bit taller). I used precisely this:



(This graph was produced to my instructions by Wolfram Alpha.) There really should be solid dots at the ends of the curve.)

5 *The population of a certain city is given approximately by*

$$C = 3t^2 + 1,$$

where C is the city's population in thousands and t is the time in years since the city was founded. The population of the city's metropolitan area is given approximately by

$$M = C + \frac{1}{8}tC,$$

where M is the metropolitan population in thousands.

a *Five years after the city was founded, how fast (at what rate) is its population growing? (Show at least what numerical calculation you make, and be sure to include appropriate units in your final answer.)*

How fast the population is growing is the derivative of population with respect to time:

$$\begin{aligned} C &= 3t^2 + 1; \\ \mathrm{d}C &= \mathrm{d}(3t^2 + 1) \\ &= \mathrm{d}(3t^2) \\ &= 3\,\mathrm{d}(t^2) \\ &= 3(2t\,\mathrm{d}t) = 6t\,\mathrm{d}t; \\ \frac{\mathrm{d}C}{\mathrm{d}t} &= 6t. \end{aligned}$$

Five years after the city was founded, $t = 5$, so $\mathrm{d}C/\mathrm{d}t = 6(5) = 30$. Since C is in thousands and t is in years, the city's population is growing at a rate of 30 000 **per year**.

b *Five years after the city was founded, how fast (at what rate) is the population of its metropolitan area growing? (Show at least what numerical calculation you make, and be sure to include appropriate units in your final answer.)*

Again, how fast the population is growing is the derivative of population with respect to time:

$$\begin{aligned}
M &= C + \frac{1}{8}tC; \\
\mathrm{d}M &= \mathrm{d}\left(C + \frac{1}{8}tC\right) \\
&= \mathrm{d}C + \mathrm{d}\left(\frac{1}{8}tC\right) \\
&= \mathrm{d}C + \frac{1}{8}\,\mathrm{d}(tC) \\
&= \mathrm{d}C + \frac{1}{8}(C\,\mathrm{d}t + t\,\mathrm{d}C) = \mathrm{d}C + \frac{1}{8}C\,\mathrm{d}t + \frac{1}{8}t\,\mathrm{d}C; \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{\mathrm{d}C}{\mathrm{d}t} + \frac{1}{8}C + \frac{1}{8}t\frac{\mathrm{d}C}{\mathrm{d}t}.
\end{aligned}$$

Five years after the city was founded, $t = 5$ and (from part a) $\mathrm{d}C/\mathrm{d}t = 30$; also, $C = 3(5)^2 + 1 = 76$ then. Thus, $\mathrm{d}M/\mathrm{d}t = (30) + 1/8\,(76) + 1/8\,(5)(30) = 58.25$. Since M is in thousands and t is in years, the city's metropolitan area's population is growing at a rate of 58 250 **per year**.

6 *Suppose that the revenue from selling x specialty items in a year is*

$$R = 8x - x^2,$$

while the cost is

$$C = 2x + 5,$$

both measured in thousands of dollars.

a *How many items should be sold in a year to maximise profit? (Show at least what numerical calculation you make.)*

The profit is the revenue minus the cost:

$$P = R - C.$$

To maximise profit, the differential of this should be zero:

$$\begin{aligned}
0 = \mathrm{d}P &= \mathrm{d}R - \mathrm{d}C; \\
\mathrm{d}C &= \mathrm{d}R; \\
\mathrm{d}(2x + 5) &= \mathrm{d}(8x - x^2); \\
2\mathrm{d}x &= 8\,\mathrm{d}x - 2x\,\mathrm{d}x; \\
2 &= 8 - 2x; \\
x &= 3.
\end{aligned}$$

Therefore, 3 **items** should be sold in a year to maximise profit.

b *How many should be sold to maximise revenue? (Show at least what numerical calculation you make.)*

To maximise revenue, its differential should be zero:

$$\begin{aligned} 0 = \mathrm{d}R &= \mathrm{d}(8x - x^2) \\ &= 8\,\mathrm{d}x - 2x\,\mathrm{d}x; \\ 0 &= 8 - 2x; \\ x &= 4. \end{aligned}$$

Therefore, 4 **items** should be sold in a year to maximise revenue.

c **Extra credit:** *How many should be sold to minimise cost? (Show at least what numerical calculation you make, or explain your reasoning.)*

I was a little sloppy in the previous parts; I should also have checked that the profit or revenue is more when its differential is zero than when $x = 0$ or as $x \to \infty$. However, this is true (which is typical for profit and revenue). In this case, I have to be careful, because $\mathrm{d}C$ is *never* zero:

$$\begin{aligned} 0 = \mathrm{d}C &= \mathrm{d}(2x + 5) \\ &= 2\,\mathrm{d}x; \\ 0 &= 2, \end{aligned}$$

which is simply false. But when $x = 0$ (the lowest possible value, since you can't sell less than nothing), $C = 2(0) + 5 = 5$; while as $x \to \infty$, $C \to 2(\infty) + 5 = \infty + 5 = \infty$. Thus the smallest possible value of C is 5, so the cost is minimised when $x = 0$, that is when **no items** are sold.