This is a summary of the concepts of integral calculus.

Definite integrals

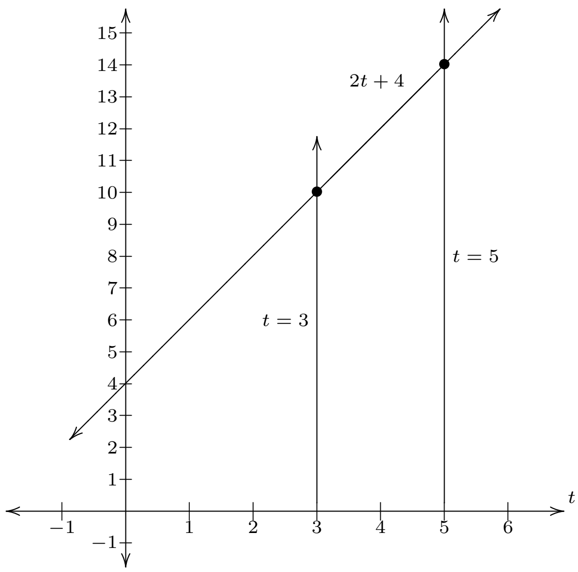
Just as the *differential* of a finite quantity is an infinitesimal (infinitely small) change in that quantity, so the **definite integral** of an infinitesimal quantity is the sum of infinitely many values of that quantity, giving a finite result. If x and y are standard smooth quantities, then $y\,\mathrm{d}x$ is a typical infinitesimal smooth quantity. If we add this up from the point where $x = a$ to the point where $x = b$, then we get the **definite integral**

$$\int_{x=a}^{x=b} y\,\mathrm{d}x.$$

As long as the same variable x is used throughout, then it's safe to abbreviate this as

$$\int_a^b y\,\mathrm{d}x.$$

For example, $\int_3^5 (2t+4)\,\mathrm{d}t$ is the sum, as t varies smoothly from 3 to 5, of the product of $2t + 4$ and the infinitesimal change in t at each stage along the way. We can think of this product as giving the area of a rectangle whose height is $2t + 4$ and whose width is $\mathrm{d}t$; if we line these rectangles up side by side, then they combine to give a trapezoid:



We can find out the area of this trapezoid using geometry, since its width is $5 - 3 = 2$ and its height varies linearly from $2(3) + 4 = 10$ to $2(5) + 4 = 14$. Therefore,

$$\int_3^5 (2t+4)\,\mathrm{d}t = \frac{10+14}{2} \cdot 2 = 24.$$

Normally, you can't evaluate an integral by drawing a picture like this; I'll come back to how we can calculate it after a brief digression.

Antidifferentials

If $y\,\mathrm{d}x = \mathrm{d}u$, then u is an **antidifferential** of $y\,\mathrm{d}x$. However, u is *not the only* antidifferential of $y\,\mathrm{d}x$; if C is any constant, then $\mathrm{d}(u+C) = y\,\mathrm{d}x$ too, so $u+C$ is also an antidifferential of $y\,\mathrm{d}x$. However, for a continuously defined quantity, there is *no other* antidifferential of $y\,\mathrm{d}x$. Antidifferentials are denoted by '$\int$', so we have

$$\int \mathrm{d}u = u + C$$

by definition. An antidifferential is also called an **indefinite integral**.

For example,

$$\mathrm{d}(t^2+4t) = 2t\,\mathrm{d}t + 4\,\mathrm{d}t = (2t+4)\,\mathrm{d}t,$$

so

$$\int (2t+4)\,\mathrm{d}t = \int \mathrm{d}(t^2+4t) = t^2+4t+C.$$

As $2t+4$ is the derivative of t^2+4t with respect to t, we also say that t^2+4t is an **antiderivative** of $2t+4$ with respect to t.

To find antidifferentials (or antiderivatives), we must run the rules for differentials (and derivatives) backwards. This is often a subtle process, which I'll return to later.

The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** relates definite and indefinite integrals. There are two parts:

1. $\mathrm{d}\left(\int_{t=a}^{b} f(t)\,\mathrm{d}t\right) = f(b)\,\mathrm{d}b - f(a)\,\mathrm{d}a$;
2. $\int_{t=a}^{b} \mathrm{d}f(t) = f(b) - f(a)$.

The first part applies whenever f is a continuous function (assuming that a and b are differentiable quantities); in particular, it claims that the integral exists and is differentiable. The second part applies whenever f is a differentiable function (assuming that t is a differentiable quantity); in particular, it claims that the integral exists.

Although both of these parts refer directly to definite integrals, indefinite integrals (antidifferentials) appear implicitly because of the presence of the differentials. Specifically, the first part claims that the definite integral that appears in it is an antidifferential of the differential form on its right-hand side, and the second part shows how to evaluate a definite integral of a differential form whose antidifferential is known.

It's the second part that's really useful. If you want to evaluate a definite integral $\int_a^b y\,\mathrm{d}x$, then you should first figure out the indefinite integral $\int y\,\mathrm{d}x$. If the answer to this is u (or rather $u+C$), then this means that $y\,\mathrm{d}x = \mathrm{d}u$; that is, u is an antidifferential of $y\,\mathrm{d}x$. Therefore, $\int_{x=a}^{x=b} y\,\mathrm{d}x = \int_{x=a}^{x=b} \mathrm{d}u$, and the FTC tells us that this is equal to $u|_{x=a}^{x=b}$ (that is, evaluate u when $x=b$ and when $x=a$, then subtract). As this last expression is simply a difference, you can figure it out using simple algebra.

For example, consider

$$\int_{t=3}^{t=5} (2t+4)\,\mathrm{d}t.$$

In the last section, we saw that $\int (2t+4)\,\mathrm{d}t = t^2+4t+C$; in other words, $(2t+4)\,\mathrm{d}t = \mathrm{d}(t^2+4t)$. Therefore,

$$\begin{aligned}\int_3^5 (2t+4)\,\mathrm{d}t &= \int_3^5 \mathrm{d}(t^2+4t) = (t^2+4t)|_3^5 \\ &= \left((5)^2+4(5)\right) - \left((3)^2+4(3)\right) = (45) - (21) = 24.\end{aligned}$$

(Notice that this is the same answer as when I did this using geometry!)

This also explains why the same term 'integral' and symbol '$\int$' are used for both the definite integral (a sum of infinitely small quantities) and the indefinite integral (the antidifferential). They at first appear to be completely different concepts, but in reality they are closely related, through the Fundamental Theorem of Calculus.

Integration techniques

This leaves us with one problem: how do we find indefinite integrals?

Each rule for differentiation gives us a rule for integration. In the table below, I have some rules for differentiation (all of which you should know by now), together with corresponding rules for integration:

$$\begin{aligned}
\mathrm{d}(u+v) &= \mathrm{d}u + \mathrm{d}v, &\quad \int (y+z)\,\mathrm{d}x &= \int y\,\mathrm{d}x + \int z\,\mathrm{d}x;\\
\mathrm{d}(ku) &= k\,\mathrm{d}u, &\quad \int ky\,\mathrm{d}x &= k\int y\,\mathrm{d}x;\\
\mathrm{d}(uv) &= v\,\mathrm{d}u + u\,\mathrm{d}v, &\quad \int u\,\mathrm{d}v &= uv - \int v\,\mathrm{d}u;\\
\mathrm{d}(u^n) &= nu^{n-1}\,\mathrm{d}u, &\quad \int u^m\,\mathrm{d}u &= \frac{1}{m+1}u^{m+1} + C;\\
\mathrm{d}(\mathrm{e}^u) &= \mathrm{e}^u\,\mathrm{d}u, &\quad \int \mathrm{e}^u\,\mathrm{d}u &= \mathrm{e}^u + C;\\
\mathrm{d}(\ln|u|) &= \frac{1}{u}\,\mathrm{d}u, &\quad \int \frac{1}{u}\,\mathrm{d}u &= \ln|u| + C;\\
\mathrm{d}(\sin u) &= \cos u\,\mathrm{d}u, &\quad \int \cos u\,\mathrm{d}u &= \sin u + C;\\
\mathrm{d}(\cos u) &= -\sin u\,\mathrm{d}u, &\quad \int \sin u\,\mathrm{d}u &= -\cos u + C;
\end{aligned}$$

etc.

Using these rules, you can work out all of the integrals in the textbook through Chapter 6, and then some.

For example, to find $\int (2t+4)\,\mathrm{d}t$:

$$\int (2t+4)\,\mathrm{d}t = \int 2t\,\mathrm{d}t + \int 4\,\mathrm{d}t = 2\int t^1\,\mathrm{d}t + 4\int \mathrm{d}t = 2\left(\frac{1}{2}t^2\right) + 4t + C = t^2 + 4t + C.$$

This is the same answer as we got before, but this time I didn't have to guess the answer and get lucky; I was able to actually calculate it.

For more complicated integrals, there are fancier techniques. Rather than learn all of these, you can program them into a computer. You can even go to `http://integrals.wolfram.com/` for a free Internet service that will do this for you!

Summary

To find the indefinite integral $\int y\,\mathrm{d}x$, you need to use integration techniques; your answer will still have the variable in it and should end with a new constant term C. To find the definite integral $\int_a^b y\,\mathrm{d}x$, first find the indefinite integral and then take a difference; assuming a and b are constants, your answer will also be constant (without the C).

So for example, to find the definite integral of $2t+4$ with respect to t from 3 to 5:

$$\int_3^5 (2t+4)\,\mathrm{d}t = \int_3^5 (2t^1\,\mathrm{d}t + 4\,\mathrm{d}t) = \left(2\left(\frac{1}{2}t^2\right) + 4t\right)\Bigg|_3^5 = (t^2+4t)|_3^5 = 45 - 21 = 24.$$

This is simply a combination of calculations that I did earlier, to find the indefinite integral and to apply the FTC.