Directions and limits

A **direction** in some variable describes not only whether the variable is increasing or decreasing (that is its literal direction on a number line) but also if there is a limiting value that it approaches but does not reach. The basic directions that we study in this course take the following four forms, where x may be any variable and c may be any constant:

- as x increases without bound: $x \to \infty$;
- as x decreases without bound: $x \to -\infty$;
- as x increases towards c: $x \to c^-$;
- as x decreases towards c: $x \to c^+$.

Any two or more of these directions may be combined, but the only type of combined direction in the textbook is this:

- as x approaches c: $x \to c$;

which is the combination of $x \to c^-$ and $x \to c^+$. That said, other combinations are also sometimes studied, especially the combination of $x \to \infty$ and $x \to -\infty$. (You can also consider fancier directions, for example as x increases without bound *while taking only integer values*, which is relevant to the material in Section 9.1 of the textbook. For now, however, I'll stick to the 5 types of directions covered in Chapter 2.)

If D is any direction and u is any variable quantity, then we indicate the value to which u approaches as change occurs in the indicated direction as

$$\lim_{D} u$$

in a diplayed equation or as $\lim_D u$ in running text. (The textbook likes to write u as $f(x)$, and this is certainly convenient when it comes to the formal definition, but in practice you'll start with an expression involving the variable x, and it's not necessary to think of this as given by a function.) We will examine the case when u approaches a real value L, as well as the case when u increases without bound or decreases without bound. In the first case, we say that the limit **converges**; in the second case, we say that the limit **diverges** to (positive or negative) infinity. (Other types of behaviour are also possible, which are also kinds of divergence, but for now I'll stick to the 3 types of results covered in the textbook.) A limit as x approaches c is one of the three kinds of results that we are considering if and only if the limits as x increases and decreases towards c are both this same result.

So in total, there are fifteen kinds of limits that we will consider, for the five kinds of directions (four basic and one combined) and the three kinds of answers:

$$\begin{array}{lll}
\lim_{x\to\infty} u = L; & \lim_{x\to\infty} u = \infty; & \lim_{x\to\infty} u = -\infty; \\
\lim_{x\to-\infty} u = L; & \lim_{x\to-\infty} u = \infty; & \lim_{x\to-\infty} u = -\infty; \\
\lim_{x\to c^-} u = L; & \lim_{x\to c^-} u = \infty; & \lim_{x\to c^-} u = -\infty; \\
\lim_{x\to c^+} u = L; & \lim_{x\to c^+} u = \infty; & \lim_{x\to c^+} u = -\infty; \\
\lim_{x\to c} u = L; & \lim_{x\to c} u = \infty; & \lim_{x\to c} u = -\infty.
\end{array}$$

To see how to read these aloud, I'll consider the last one as an example; this says that the **limit**, as x approaches c, of u is negative infinity.

It's sometimes convenient to think of ∞ and $-\infty$ as numbers like the real number c or L, only numbers of an infinite magnitude. Similarly, it's sometimes convenient to think of c^+ and c^- as numbers that are infinitely close to (but distinct from) the real number c. Then the meanings of the directions are as follows:

- $x \to \infty$: what happens when x is positive and infinite?
- $x \to -\infty$: what happens when x is negative and infinite?
- $x \to c^-$: what happens when x is infinitely close to but less than c?
- $x \to c^+$: what happens when x is infinitely close to but greater than c?
- $x \to c$: what happens when x is infinitely close to but distinct from c?

Similarly, the meanings of the results are as follows:

- $\lim_D u = L$: u is equal to or infinitely close to L;
- $\lim_D u = \infty$: u is positive and infinite;
- $\lim_D u = -\infty$: u is negative and infinite.

This can be made into a rigorous definition, by extending the real number system to the *hyperreal* number system, although this is not the basis of the definition that we will be using. Still, it can be useful for intuition.