One-variable Calculus

Toby Bartels

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Welcome to Calculus! Here are my supplemental notes for one-variable Calculus, giving alternative ways to think about some things, practical advice, and sometimes more theoretical detail.

This does not cover everything that you need to know (although it covers a lot); you should also have the official course textbook, which is the 3rd Edition of *University Calculus: Early Transcendentals* by Hass et al published by Addison–Wesley (Pearson). There are also some references in these notes to that textbook.

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1 Preliminaries

Before beginning this class, you should be familiar with the basic algebraic properties of real numbers.

By default, all of the numbers that we work with will be real numbers. (Most of Calculus applies just as well to complex numbers, but a complete understanding of Calculus in even one complex variable requires some ideas from multivariable Calculus, which these notes do not cover.) In particular, if a is a negative number, then $\sqrt[n]{a}$ is undefined when n is an even integer and negative when n is an odd integer. More generally, if a is a negative number, then a^p is defined *only* if p is a rational number whose denominator in lowest terms is odd; in this case, a^p is positive if the numerator of p is even and negative if the numerator of p is odd. Note that $(a^2)^{1/2} = \sqrt{a^2} = |a|$, while $a^{2 \cdot 1/2} = a^1 = a$, which is different when a is negative, so the rule that $(a^x)^y = a^{xy}$ does not hold in general (although it does hold when a is a positive number).

Although 0^x is undefined whenever x is negative (because this amounts to dividing by zero), we need to define $0^0 = 1$ in order to make some formulas work correctly. Although the textbook says that 0^0 is undefined, this contradicts some things that that book says about polynomials and power series. (Section 9.7 of the official textbook, beginning with the definition of power series on page 523, is the first place where this is important; see also the discussion of power series starting on page 45 in these notes.) It's possible to take a more nuanced approach, where 0^x is 1 when x is an *integer*-valued variable with the value 0 while 0^x is undefined when x is a *real*-valued variable with the value 0; however, this makes the meaning of 0^0 ambiguous without context, so I prefer to simply say that $0^0 = 1$. Nevertheless, this will require some care when it comes to rules for evaluating limits.

The main difference between my approach to Calculus and the textbook's is that I make more use of *differentials*. Calculus was originally developed using differentials, and many calculations are easier to do this way. Furthermore, differentials are often used in applications, especially (but not only) to physics. They fell out of fashion with mathematicians towards the end of the 19th century, when Calculus was first put on a rigorous logical foundation, because this foundation did not include differentials. However, a rigorous logical development of differentials as well had been achieved by the middle of the 20th century, so there is no longer any reason to avoid them. You can do almost everything with the textbook's methods if you want, but I encourage you to try using differentials.

1.1 *Functions*

Another difference between these notes and the textbook is that I will never be sloppy with function notation.

In an expression such as

$$y = f(x),$$

the variables x and y stand for real numbers, while the variable f stands for a function. (Usually this variable is actually a constant, because f always refers to the same function throughout the problem, although there can also be situations where the function itself is allowed to vary.) A function is not a number but rather a process for turning one number into another. When speaking of specific numbers, this is usually not a problem; for example, $f(2) = 4$ means that the function f is a process that (among other things) turns the number 2 into the number 4.

The statement that $f(x) = x^2$ is more ambiguous; in a context where the variable x already appears, this means that the function f is a process that turns the number x (whatever number that is) into the number x^2. But in a context where x does not already have a meaning, this statement usually means that the function f is a process that turns *every* real number into its square, which is a complete description of the function. In this case, it is better to say something like

$f(x) = x^2$ for all x,

and I will usually say something like this.

Another way to completely describe this function is to write

$$f = (x \mapsto x^2).$$

This is analogous to defining a set S as $S = \{x \mid x > 2\}$; in each case, you introduce a new *dummy variable* and then you either give an expression (to define a function) or else you give an equation, inequality, or other statement (to define a set), in each case using that dummy variable. You can even do this without giving the function (or set) a name, by (for example) just referring to the function $(x \mapsto x^2)$ or the set $\{x \mid x > 2\}$. Although the textbook does this with sets, it never does this with functions; so in order to avoid burdening you with additional notation to learn, I will not do this either. It can be very handy, however.

The real problem is when the same symbol is used both to refer to a function and to its output value, as in

$$A = A(x),$$

which you might see (for example) in a problem in which the *a*rea of some shape depends on something else. I will never do this! Either I will use A to refer to the area itself, or I will use A to refer to the function that indicates how this area depends on x, but I will not use the same symbol for both of these. If I need to refer to both of these, then I will use two different symbols. Most of the time, however, it's enough to have a symbol for the area itself and to leave the function unnamed. (The notation for evaluation described on page 4 can help with this.)

When we cover derivatives later on, you will learn various symbols used for this concept; and when $y = f(x)$, then I will also write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

(What this means is explained on page 13.) The textbook will sometimes write y' or $\mathrm{d}f/\mathrm{d}x$ in this situation, but I never will, and this is important to ensure that the ordinary rules of algebra continue to apply to such expressions. (For example, you can multiply both sides of the equation above by $\mathrm{d}x$ to get $\mathrm{d}y = f'(x)\,\mathrm{d}x$, which would be difficult to do correctly using the wrong symbols.) I will not count it against you if you are as sloppy as the textbook about this, because I don't think that it's fair to require more of you than the textbook writers can manage; however, if you get confused by your notation and make a mistake, then that will count against you! So I encourage you to use precise notation.

1.2 *Variables*

In Calculus, we study *variable* quantities, that is quantities whose values may vary (or change).

In Algebra, we often use the word 'variable' to refer to any quantity whose value we don't know, even if this value is fixed and never changes throughout the problem. In fact, the standard Algebra problem, solving an equation such as $2x + 3 = 5$, involves figuring out the value of the variable; so it had only one value all along, and we just had to figure out what it was. So if x is a variable in an Algebra problem, and at some point we decide that the value of x is 1, then this may well mean that x is 1 throughout the entire problem. (That's not always the case in Algebra, but it often is.)

In Calculus, we take the word 'variable' more seriously. If x is a variable in a Calculus problem, then x might be 1 at some point, but it will probably be 6 at some other point in the problem. (And more often than not, it will take all of the values in between 1 and 6 along the way, such as $1\frac{1}{2}$, π, and 5.789.) Furthermore, if x and y are two variables that appear in the same problem, then the value of y will usually change as the value of x changes. Calculus is primarily about exactly this sort of thing: *how* one quantity changes as another quantity changes.

In the simplest cases, it turns out that y is a function of x; that is, there is a fixed function f such that $y = f(x)$ remains true as x and y vary. Calculus textbooks generally try to fit everything into this mould, but it doesn't always come out like this naturally. Often, you know that both x and y are changing, but it's not obvious that the value of x at some point is enough information to figure out the value of y at that point; yet when you write $y = f(x)$, you're assuming that it is enough information.

Most of the time, however, we can assume that there is some variable t, called the *independent variable*, such that every other variable in the problem is a function of t. That is, if x and y appear in the problem, then there are fixed functions f and g such that $x = f(t)$ and $y = g(t)$ throughout the problem. (Then x and y are called *dependent variables*, since their values depend on the values of t, through the functions f and g.) But this variable t might not show up directly! Calculus books will usually tell you (especially in word problems) that it's necessary to pick an independent variable from among the variables that appear in the problem, but really it's enough to informally visualize the range of variation in the problem, and you can treat all of the variables on an equal footing. All the same, for the sake of formal definitions, we will assume that there is an independent variable t and that every other variable is a function of it, even though in practice we don't have to identify it. (Of course, you don't have to call the independent variable 't', but I usually will, just to have a consistent name.)

If we're not going to refer directly to t, then we're not going to refer directly to f and g either, so we need some way to refer to the values of these functions without referring to the functions themselves. Here is how we do it formally:

If $u = f(t)$, then $u|_{t=c} = f(c)$.

(This is called *evaluation notation*.) More generally, if P is some statement that is only true once, then P implies the statement $t = c$ for some value of c, so we can make sense of $u|_P$. Even if P is a statement that might not only be true once, as long as every possible value of $u|_P$ is the same, then we can still make sense of $u|_P$. Finally, even if there are different possible values of $u|_P$, then the value of $u|_P$ still varies, but at least it doesn't vary as much as u itself, since there are now fewer possibilities.

This all sounds very abstract (because it is), but the concrete application is straightforward; here are some examples:

$$\begin{aligned} x|_{x=5} &= 5, \\ (2x+3)|_{x=4} &= 2(4)+3 = 11, \\ (2x+3y)|_{\substack{x=4,\\ y=5}} &= 2(4)+3(5) = 23. \end{aligned}$$

Taking the last of these for example, there is no need to think about what t is when $x = 4$ and $y = 5$, and indeed without considering how x and y depend on this unspecified independent variable t, the value of t is impossible to know. Nevertheless, we know that no matter what t may be, if $x = 4$ and $y = 5$ at that value of t, then $u = 2x + 3y$ is definitely $2(4) + 3(5) = 23$ at that same value of t, and that is enough. So all that you have to do in practice is to plug in the given values and perform the given calculation.

Sometimes (generally only in the middle of a problem or in something theoretical) you can't work out the value completely; for example,

$$(2x+3y)|_{x=4} = 2(4) + 3(y|_{x=4}) = 8 + 3y|_{x=4}.$$

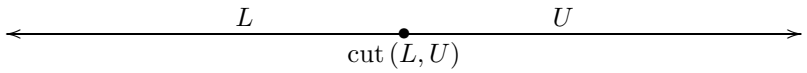
If we don't know anything more about the relationship between x and y, then we don't know the value of y when $x = 4$, so this is all that we can say in this example, but at least we were able to work out part of it.

1.3 *Real numbers*

In this course, we work with the real numbers, which are supposed to correspond to points on a number line. Ultimately, all of the properties of real numbers derive from intuitive geometric properties of points on a line. For example, the arithmetic operations of addition, subtraction, multiplication, and division can be defined in terms of changes of position and scale on the number line. The order relation between real numbers ($<$ and $>$) also derives from relative position on a line. (You have to specify the numerical values of at least two points, such as 0 and 1, in order to make a geometric line into a number line, but once you have those two points, then everything else follows.)

The most advanced of the fundamental properties of the number line is its *completeness*. There are many ways to express completeness, but my favourite is this:

If you pick out two nonempty regions of the number line, one on the left called L and one on the right called U, which don't overlap but otherwise cannot be extended further, then there is a single point between them, called $\text{cut}\,(L,U)$, the *cut* between L and U.



We can make this logically precise (in terms of the order relation on real numbers): Suppose that L and U are two sets of real numbers (making precise what regions of the number line are), with these properties:

- There is some $r \in L$ and some $s \in U$ (which is what it means for L and U to be nonempty);
- If $r \in L$ and $s \in U$, then $r < s$ (which is what it means for L to be on the left and U on the right without overlapping);
- If $r < s$, then $r \in L$ or $s \in U$ (which is what it means to say that L and U cannot be extended further).

(Note that 'or' in math, as here, normally includes the possibility of both.) Then there exists a real number $\text{cut}\,(L,U)$ with this property:

- If $r \in L$ and $s \in U$, then $r \leq \text{cut}\,(L,U) \leq s$ (which is what it means for $\text{cut}\,(L,U)$ to be between L and U).

A couple more important properties follow from what was said above:

- The number $\text{cut}\,(L,U)$ is the *only* real number between L and U;
- If $r < \text{cut}\,(L,U) < s$, then $r \in L$ and $s \in U$.

The point of all this is to be able to prove that a real number exists. For example, in order to prove rigorously that every real number c has a cube root $\sqrt[3]{c}$ (and has anybody ever showed you why this is true or did you just take it on faith?), you first define L as $\{x \mid x^3 < c\}$ and U as $\{x \mid x^3 > c\}$, check that L and U have the necessary properties listed above (which takes a bit of work with algebra), conclude that $\text{cut}\,(L,U)$ exists with the properties listed above, and check (using those properties) that $\text{cut}\,(L,U)^3 = c$ (which takes a lot more work with algebra). Thus, this cut is the cube root $\sqrt[3]{c}$.

This method of proving that a real number exists is also practical, because it shows us how to approximate its value as closely as we like. For example, to approximate $\sqrt[3]{2}$ to 4 decimal places, you look at some nearby possibilities, such as $1.0001, 1.0002, 1.0003, \ldots, 1.9997, 1.9998, 1.9999$. Somewhere in this list are two numbers right next to each other, one of which has a cube less than 2 (so it's in L) and one of which has a cube greater than 2 (so it's in U). Then we approximate $\sqrt[3]{2}$ to 4 decimal places by saying that it's in between these two numbers. (As it happens, these two numbers are 1.2599 and 1.2600; also, $1.25995^3 > 2$, so $\sqrt[3]{2}$ rounds to 1.2599.) There are more efficient ways to calculate cube roots (such as Newton's Method, described on page 23), but this proof that they exist at least gives *one* way to calculate them, to start with.

2 Limits and continuity

There are four main operations considered in Calculus: limits, derivatives (or differentials), integrals (or antidifferentials), and sums of infinite series. (The last of these is only covered in Calculus 2.) Here we will look at the first one: limits. These are also closely related to the concept of continuity, which is actually the easiest concept to define.

2.1 *Continuity*

In Calculus, we not only study variable quantities; we study quantities that are *continuously* varying. This implies in particular that a quantity y that varies from 1 to 6 will pass through $1\frac{1}{2}$, π, and 5.789, and everything else in between.

In real life, we can never measure or fix the value of a such a quantity y exactly, down to the last decimal place; after all, there are infinitely many decimal places, but we can only do a finite amount of work. So, it is key to the study of real numbers that we can *approximate* them to any finite number of decimal places (among other ways). That is what the stuff about cuts on page 5 accomplishes.

Also in Calculus, we study how one quantity y varies along with another quantity x. This may be expressed by saying that y is a *function* of x; if f is the function, then $y = f(x)$. But in practice, we only know x and y *approximately*, so if we only use an approximate value of x, then $f(x)$ should still be an approximate value of y. For example, suppose that $f(x) = x^2$ for all x; if you know that x is approximately 2, then you know that $y = f(x)$ is approximately $2^2 = 4$.

This doesn't work with every function! For example, suppose that g is the piecewise-defined function

$$g(x) = \begin{cases} x+1 & \text{for } x < 2, \\ x+3 & \text{for } x \geq 2; \end{cases}$$

if you only know that x is approximately 2, then you really don't know if $g(x)$ is approximately $2 + 1 = 3$ or approximately $2 + 3 = 5$. Of course, if you knew that x is *exactly* 2, then you would know that $g(x)$ is $2 + 3 = 5$; but it's no good if you only know x approximately.

In these examples, we say that g has a **discontinuity** at 2, while f is **continuous** at 2. (In fact, f is continuous everywhere, while g is continuous everywhere except at 2.) So the idea is this:

> A function f is **continuous** at a real number c if, whenever $x \approx c$ (meaning that x is approximately equal to c), $f(x) \approx f(c)$.

So if you only know that $x \approx c$, then that's enough information to know $f(x)$ approximately (specifically, that $f(x) \approx f(c)$).

Actually, we should take care about where f is defined. Sometimes Calculus textbooks say that f has a discontinuity at c if f is undefined at c (that is if $f(c)$ does not exist), and sometimes they don't; but in any case, f is not continuous there: f must be defined first in order to be continuous. On the other hand, if f is undefined at x, then we don't hold that against f; for example, we want to say that $f(x) = \sqrt{x}$ is continuous at 0, even though $f(x)$ does not exist (as a real number) whenever $x < 0$. So a more careful definition is this:

> A function f is **continuous** at a real number c if $f(c)$ exists and, whenever $x \approx c$ and $f(x)$ exists, $f(x) \approx f(c)$.

This is still not a completely rigorous definition, because it doesn't explain how close we need to be to say that one quantity is approximately equal to another. (Basically, the answer is this: as close as you need, and as close as you want.) But I will save that for a bit later. Already, this basic idea should be enough to allow you to judge continuity of a function from its graph.

To judge continuity of a function from a formula, it's convenient to know that any function is continuous (wherever it is defined) if it has a formula that uses only these operations: addition, subtraction, multiplication, division, absolute values, opposites, reciprocals, raising to powers when the exponent is constant or the base is always positive, extracting roots when the index is constant or the radicand is always

positive, logarithms, trigonometric functions, and inverse trigonometric functions. These are pretty much all of the functions that you ever deal with!

So, the exceptions in practice are much rarer: exponentiation where the exponent varies and the base can be zero or negative, roots where the index varies and the radicand can be zero or negative, and piecewise-defined functions. Of these, only piecewise-defined functions are likely to come up. These functions *can* be continuous, but only if the values agree on both sides whenever two pieces join. So for example, while

$$g(x) = \begin{cases} x+1 & \text{for } x < 2, \\ x+3 & \text{for } x \geq 2 \end{cases}$$

has a discontinuity at $x = 2$,

$$h(x) = \begin{cases} x+1 & \text{for } x < 2, \\ 5-x & \text{for } x \geq 2 \end{cases}$$

is continuous at $x = 2$ (and so everywhere), because $2 + 1 = 5 - 2$.

Returning to the meaning of continuity, how close of an approximation is close enough? The key to the answer is that a real number may be approximated as precisely as you wish, as long as you put enough work into it. So for f to be continuous at c, we should be able to demand that $f(x)$ and $f(c)$ be as close together as we like (as long as we still allow for a positive distance between them). But in order to achieve that result, it's fair in turn to demand that x be as close to c as necessary (again as long as we still allow the distance to be positive). The distance between two numbers is given by subtracting and taking the absolute value, so we need to be able to ensure that $|f(x) - f(c)|$ is as small as we want (but positive) by making $|x - c|$ as small as we need (but positive).

The traditional symbols for these small but positive distances are the Greek letters 'ϵ' (lowercase Epsilon) and 'δ' (lowercase Delta). For this reason, this is sometimes called the ϵ-δ (or epsilon-delta) definition; this method is also called *epsilontics*. So here is the rigorous definition:

> A function f is **continuous** at a real number c if $f(c)$ exists and, for each positive number ϵ (no matter how small), there is some positive number δ (possibly quite small), such that whenever $|x - c| < \delta$ and $f(x)$ exists, $|f(x) - f(c)| < \epsilon$.

This is fairly complicated, but you can view it as a game, involving a function f and a number c such that $f(c)$ exists.

- I challenge you with a positive number ϵ.
- You respond with a positive number δ.
- I reply with a value of x such that $|x - c| < \delta$ and $f(x)$ exists.
- You win if $|f(x) - f(c)| < \epsilon$.

If you can win this game, no matter what choices I make, then f is continuous at c. On the other hand, if I can win no matter what choices you make, then f has a discontinuity at c.

To see how this matters in practice, suppose again that $f(x) = x^2$ for all x and you're told that $x \approx 2$; you want to judge how precisely you know that $x^2 \approx 4$. To be specific, suppose that you want to be guaranteed that x^2 rounds to 4 to at least 3 digits after the decimal point, in other words that $|x^2 - 4| < \frac{1}{2} \times 10^{-3}$. (That is, ϵ is $\frac{1}{2} \times 10^{-3} = 0.0005$.) This means that you want x^2 to be between $4 - \frac{1}{2} \times 10^{-3} = 3.9995$ and $4 + \frac{1}{2} \times 10^{-3} = 4.0005$. Taking square roots (and assuming that x is positive, since it's near 2), this means that x is between $\sqrt{3.9995} \approx 1.999\,87$ and $\sqrt{4.0005} \approx 2.000\,12$. To be really sure that this is true, round up the lower number and round down the upper number: x should be between 1.9999 and 2.0001. Subtracting these from 2, this means that $|x - 2| < 0.0001$. (That is, δ is 0.0001; if the upper and lower estimates give you different values of δ, then use the smaller one to be safe.) So if you can verify that x is at least *that* close to 2, then you can be confident that x^2 is at least as close to 4 as you needed. (That f is continuous at 2 means that no matter how precisely you need to know that $x^2 \approx 4$, you'll be able to perform a calculation like this, at least in principle, to find out how precisely you need to require that $x \approx 2$.)

2.2 *Directions*

A **direction** in some variable describes not only whether the variable is increasing or decreasing (that is its literal direction on a number line) but also if there is a limiting value that it approaches but does not reach. The basic directions that we study in this course take the following four forms, where x may be any variable and c may be any constant:

- as x increases without bound: $x \to \infty$;
- as x decreases without bound: $x \to -\infty$;
- as x increases towards c: $x \to c^-$;
- as x decreases towards c: $x \to c^+$.

Any two or more of these directions may be combined, but the only type of combined direction in the textbook is this:

- as x approaches c: $x \to c$;

which is the combination of $x \to c^-$ and $x \to c^+$. That said, other combinations are also sometimes studied, especially the combination of $x \to \infty$ and $x \to -\infty$, which is written $x \to \pm\infty$. (You can also consider fancier directions, for example as x increases without bound *while taking only integer values*, which is relevant to the material in Section 9.1 of the textbook. For now, however, I'll stick to the 5 types of directions covered in Chapter 2.)

It's sometimes convenient to think of ∞ and $-\infty$ as numbers like the real number c, only numbers of an infinite magnitude. Similarly, it's sometimes convenient to think of c^+ and c^- as numbers that are infinitely close to (but distinct from) the real number c. Then the meanings of the directions are as follows:

- $x \to \infty$: what happens when x is positive and infinite?
- $x \to -\infty$: what happens when x is negative and infinite?
- $x \to c^-$: what happens when x is infinitely close to but less than c?
- $x \to c^+$: what happens when x is infinitely close to but greater than c?
- $x \to c$: what happens when x is infinitely close to but distinct from c?

This can be made rigorous, by extending the real number system to the *hyperreal* number system, although this is not the basis of the definitions that we will be using. But in any case, it can be useful for intuition.

Ultimately, the important thing about a direction is what happens *eventually* as you move in that direction. So for example, as $x \to \infty$, it is eventually true that $x > 0$, that $x > 1$, that $x > 2$, and so on. Besides that ... well, that's it, really. If any statement P is true as $x \to \infty$, then it's true because there is some fixed number M (which you may assume is a whole number, although you don't have to do this) such that P is true whenever $x > M$. For example, $x^2 > 4$ as $x \to \infty$, because $x^2 > 4$ whenever $x > 2$. (It's also true that $x^2 > 4$ whenever $x < -2$, but that's irrelevant.)

Similarly, P is true (eventually) as $x \to -\infty$ if there is some number M such that P is true whenever $x < -M$. Also, P is true in the combined direction $x \to \pm\infty$ if it is true both as $x \to \infty$ and as $x \to -\infty$, in other words if there is some number M such that P is true whenever $|x| > M$. Next, P is true as $x \to c^+$ if there is some positive number ϵ (which you may assume is $1/M$ for some natural number M, although you don't have to do this) such that P is true whenever $c < x < c + \epsilon$; and P is true as $x \to c^-$ if there is some positive number ϵ such that P is true whenever $c - \epsilon < x < c$. Finally, P is true as $x \to c$ if it is true both as $x \to c^+$ and as $x \to c^-$, in other words if there is some positive number ϵ such that P is true whenever $c - \epsilon < x < c + \epsilon$ but $x \neq c$ (or equivalently whenever $0 < |x - c| < \epsilon$).

For example, $x - 2 \neq 0$ as $x \to 2$, precisely because of the $x \neq 2$ bit; the point of $x \to 2$ is that x is *close* to 2 but still *distinct* from 2. You can't say that $x - 2 > 0$ as $x \to 2$, but at least $(x-2)^2 > 0$; also, $x - 2 > 0$ as $x \to 2^+$. This sort of analysis allows you to simplify things as you work in particular directions.

2.3 *Limits*

If D is any direction and u is any variable quantity, then we indicate the value to which u approaches as change occurs in the indicated direction as

$$\lim_D u$$

in a diplayed equation or as $\lim_D u$ in running text. (The textbook likes to write u as $f(x)$, and this is certainly convenient when it comes to the formal definition, but in practice you'll start with an expression involving the variable x, and it's not necessary to think of this as given by a function.) We will examine the case when u approaches a real value L, as well as the case when u increases without bound or decreases without bound. In the first case, we say that the limit **converges**; in the second case, we say that the limit **diverges** to (positive or negative) infinity. Other types of behaviour are also possible, which are also kinds of divergence, but I won't try to analyse those now.

A limit as $x \to c$ is one of the three kinds of results that we are considering if and only if the limits as $x \to c^+$ and as $x \to c^-$ are both this same result. So in total, there are fifteen kinds of limits that we will consider, for the five kinds of directions (four basic and one combined) and the three kinds of results:

$$\begin{array}{lll}
\lim\limits_{x\to\infty} u = L; & \lim\limits_{x\to\infty} u = \infty; & \lim\limits_{x\to\infty} u = -\infty; \\
\lim\limits_{x\to-\infty} u = L; & \lim\limits_{x\to-\infty} u = \infty; & \lim\limits_{x\to-\infty} u = -\infty; \\
\lim\limits_{x\to c^-} u = L; & \lim\limits_{x\to c^-} u = \infty; & \lim\limits_{x\to c^-} u = -\infty; \\
\lim\limits_{x\to c^+} u = L; & \lim\limits_{x\to c^+} u = \infty; & \lim\limits_{x\to c^+} u = -\infty; \\
\lim\limits_{x\to c} u = L; & \lim\limits_{x\to c} u = \infty; & \lim\limits_{x\to c} u = -\infty.
\end{array}$$

To see how to read these aloud, I'll consider the last one as an example; this says that the **limit**, as x approaches c, of u is negative infinity.

If you think of ∞ and $-\infty$ as numbers of an infinite magnitude, then the meanings of the results are as follows:

- $\lim_D u = \infty$: u is positive and infinite;
- $\lim_D u = -\infty$: u is negative and infinite;
- $\lim_D u = L$: u is infinitely close to L.

This can be made into a rigorous definition of limits using the hyperreal number system, but we will only use it for intuition.

There are some alternative notations for limits that are worth knowing. First of all, instead of writing $\lim_D u$, you can also write $u|_D$, analogous to evaluation notation (page 4). That is, $u|_{x=c}$ means whatever u *equals* when x *equals* c, while $u|_{x\to c}$ means whatever u *approaches* (or equals) when x *approaches* (but is still distinct from) c.

The point of a continuous function is that these are the same; that is, f is continuous at c if and only if $f(x)|_{x=c}$ and $f(x)|_{x\to c}$ both exist and are equal. Of course, instead of writing $f(x)|_{x=c}$, you could just write $f(c)$; similarly, instead of writing $f(x)|_{x\to c}$, there is yet another notation for this:

$$f(c^{\pm}) = f(x)|_{x\to c} = \lim_{x\to c} f(x).$$

You can read this as 'f of c plus or minus'; the idea behind 'plus or minus' here is the same as in the English phrase 'more or less', meaning 'approximately', because we're looking at values of f *near* c rather than *at* c. Then f is continuous at c if and only if $f(c^{\pm}) = f(c)$ (including that these both exist).

The analogous notations for the other types of directions are $f(c^-)$, $f(c^+)$, $f(\infty)$, and $f(-\infty)$. Since things like c^+ and ∞ aren't real numbers, there should be no confusion between this function-limit notation and the usual function-evaluation notation $f(c)$. Since all of these alternative notations for limits aren't in the textbook, I won't use them, but they are good to know; they are short and handy, and you may see them elsewhere.

2.4 *Defining limits*

The simplest type of limit to define is $\lim_{x\to c} f(x)$. Note that this just depends on the function f and the real number c, which is especially clear using the notation $f(c^{\pm})$ above. If f is continuous at c, then this is supposed to be $f(c)$. But what if f is undefined or discontinuous at c?

Given a real number L, let $f_{c\mapsto L}$ be the piecewise-defined function given by

$$f_{c\mapsto L}(x) = \begin{cases} f(x) & \text{for } x \neq c, \\ L & \text{for } x = c. \end{cases}$$

That is, $f_{c\mapsto L}$ is almost the same function as f, except that $f_{c\mapsto L}(c) = L$, regardless of what $f(c)$ is (or even whether $f(c)$ exists in the first place). Now here is the definition of the limit:

> If there is a unique real number L such that $f_{c\mapsto L}$ is continuous at c, then L is the limit of f approaching c.

Note that the limit is undefined if either there is no L that makes $f_{c\mapsto L}$ continuous or if there is more than one L that makes it continuous. But that second possibility is very rare; it only happens if f is undefined approaching c, that is if f is not defined anywhere near c (in which case $f_{c\mapsto L}$ is continuous at c no matter what L is, because there is nothing nearby to compare to).

What if the limit is some kind of infinity? We can't talk about $f_{c\mapsto\infty}$, because then $f_{c\mapsto\infty}(c)$ would have to be ∞, which is not a real number. However, if $f(x)$ is increasing without bound, then $1/f(x)$ should be approaching 0. This *almost* allows us to define when the limit is ∞; the only problem is that $1/f(x)$ still approaches 0 even if $f(x)$ *decreases* without bound as well. Still we can say that

$$\lim_{x\to c} f(x) = \pm\infty \text{ if } \lim_{x\to c}\left(\frac{1}{f(x)}\right) = 0.$$

To finish the definitions that we want, we need to specify the sign of $f(x)$ as well:

$$\lim_{x\to c} f(x) = \infty \text{ if } \lim_{x\to c}\left(\frac{1}{f(x)}\right) = 0 \text{ and } f(x) > 0 \text{ as } x \to \infty;$$

$$\lim_{x\to c} f(x) = -\infty \text{ if } \lim_{x\to c}\left(\frac{1}{f(x)}\right) = 0 \text{ and } f(x) < 0 \text{ as } x \to \infty.$$

You an also define things like $\lim_{x\to c} u = L^-$ and $\lim_{x\to c} u = L^+$ by similar restrictions, but we won't be doing that.

Finally, for the general defintion of $\lim_D u$, where D is any direction and u is any expression, suppose (like I did back on the top of page 4) that x and u are both functions of some independent variable t, where x is the variable that appears in the direction D. To be precise, suppose that $u = f(t)$ and $x = g(t)$. If the direction D consists of some additional condition on the variable x, then assume that this condition holds for every value of the function g. (So for $x \to c^-$, suppose that $g(t) < c$ always, and for $x \to c^+$, suppose that $g(t) > c$ always; even for $x \to c$, still suppose that $g(t) \neq c$ always.) Then if the limit of $f(t)$ has the same value (a real number L, ∞, or $-\infty$) whenever the limit of $g(t)$ is the value given by the direction D (a real number c, ∞, or $-\infty$), then that value for the limit of $u = f(t)$ is the limit $\lim_D u$.

This definition covers much more general cases than the textbook's; for example, $\lim_{x\to 0}(\pm x) = 0$, because whenever $f(t) = \pm g(t)$ and $\lim g(t) = 0$, then $\lim f(t) = 0$. Intuitively, this should be obvious, since $\pm x \approx 0$ whenever $x \approx 0$, no matter whether it's $+x$ or $-x$. But the textbook can't make sense of this, technically, since $\pm x$ is not a function of x. The formal definition of the Riemann integral is another case where the textbook technically cannot write it down but I can.

The textbook defines limits directly using epsilontics (which is very similar to the epsilontic definition of continuity but slightly more complicated), then defines continuity using limits; I have defined continuity using epsilontics and defined limits using continuity. Our definitions come in different orders, but they are equivalent (at least in the cases where the book gives a defintion at all). In any case, the most important method of calculating limits is this:

> If f is continuous at c, then $\lim_{x\to c} f(x) = f(c)$.

This fact makes *most* limits trivial to calculate; but it's the exceptions where all of the interesting stuff happens!

For example, let g be the piecewise-defined function from page 6:

$$g(x) = \begin{cases} x+1 & \text{for } x < 2, \\ x+3 & \text{for } x \geq 2; \end{cases}$$

consider the limits of $g(x)$ in various directions. Since g is continuous everywhere except at 2, it follows that $\lim_{x \to c} g(x)$ is simply $g(c)$ for every real number c other than 2. There are still a few interesting limits of $g(x)$, however: the limits as $x \to 2^+$, as $x \to 2^-$, as $x \to \infty$, and as $x \to -\infty$. The first of these is $g(2) = 5$, basically because $g(x)$ uses the same formula when $x = 2$ as when $x > 2$; formally, it's because $x + 3$ for $x \geq 2$ is continuous as a function of x. (You can say that g is *right-continuous* at 2.) The next one, the limit as $x \to 2^-$, is 3, even though $g(2) \neq 3$ (so g is *not* left-continuous at 2). But the reason for this limit is essentially the same as the reason for the previous limit; it is that $x + 1$ for $x \leq 2$ is continuous as a function of x. Next, the limit as $x \to \infty$ is ∞, because if x is positive as $1/x \to 0$, then $x + 3$ is positive and $1/(x + 3) \to 0$, or going down to an even more basic level, because $1/(1/t + 3)$ simplifies to $t/(1 + 3t)$, which is continuous, positive when t is positive, and 0 when t is 0. Finally, the limit as $x \to -\infty$ is $-\infty$, for essentially the same reason, but now using $1/(1/t + 1)$ and looking at negative values. (This time, $1/(1/t + 1)$ can be positive even when t is negative, but not when t is sufficiently close to 0, which is what matters.)

2.5 *Calculation techniques*

Here I discuss the practical aspects of calculating limits.

The first fact to know about calculating limits is that the limit of the variable itself is already given by the direction:

$$\lim_{x \to c^-} x = c, \ \lim_{x \to c^+} x = c, \ \lim_{x \to c} x = c, \ \lim_{x \to \infty} x = \infty, \ \lim_{x \to -\infty} x = -\infty.$$

A similarly important principle is that the limit of a constant, in *any* direction, is that constant:

- $\lim_{D} C = C$. Of course, we rarely bother with limits as simple as these! However, we have the powerful principle that if an expression is built using only the usual operations,* then the limit of the expression may be computed using these operations.

---

* Addition, subtraction, multiplication, division, absolute values, opposites, reciprocals, raising to powers when the exponent is constant or the base is always positive, extracting roots when the index is constant or the base is always positive, logarithms, trigonometric operations, and inverse trigonometric operations, the same as the list of continuous operations spanning pages 6 and 7

Explicitly, each of these equations is true whenever the right-hand side is defined (so that in particular the left-hand side is automatically also defined), so long as n is constant and $\lim_D w$ is positive:

$$\lim_D (u+v) = \lim_D u + \lim_D v; \qquad \lim_D (u-v) = \lim_D u - \lim_D v;$$
$$\lim_D (uv) = \lim_D u \cdot \lim_D v; \qquad \lim_D (u/v) = \frac{\lim_D u}{\lim_D v};$$
$$\lim_D (-u) = -\lim_D u; \qquad \lim_D (|u|) = \left|\lim_D u\right|;$$
$$\lim_D (u^n) = \left(\lim_D u\right)^n; \qquad \lim_D (w^u) = \left(\lim_D w\right)^{\lim_D u};$$
$$\lim_D (\sqrt[n]{u}) = \sqrt[n]{\lim_D u}; \qquad \lim_D (\log_v u) = \log_{\lim_D v}\left(\lim_D u\right);$$
$$\lim_D (\sin u) = \sin\left(\lim_D u\right); \qquad \lim_D (\cos u) = \cos\left(\lim_D u\right);$$
$$\lim_D (\tan u) = \tan\left(\lim_D u\right); \qquad \lim_D (\cot u) = \cot\left(\lim_D u\right);$$
$$\lim_D (\sec u) = \sec\left(\lim_D u\right); \qquad \lim_D (\csc u) = \csc\left(\lim_D u\right);$$
$$\lim_D (\operatorname{asin} u) = \operatorname{asin}\left(\lim_D u\right); \qquad \lim_D (\operatorname{acos} u) = \operatorname{acos}\left(\lim_D u\right);$$
$$\lim_D (\operatorname{atan} u) = \operatorname{atan}\left(\lim_D u\right); \qquad \lim_D (\operatorname{acot} u) = \operatorname{acot}\left(\lim_D u\right);$$
$$\lim_D (\operatorname{asec} u) = \operatorname{asec}\left(\lim_D u\right); \qquad \lim_D (\operatorname{acsc} u) = \operatorname{acsc}\left(\lim_D u\right).$$

In this way, we can evaluate most limits.

We can do even more limits if we extend arithmetic to the values $\pm\infty$ as follows, where a is (in general) any real number or $\pm\infty$:

- $a + \infty = \infty$ if $a > -\infty$;
- $a - \infty = -\infty$ if $a < \infty$;
- $a \cdot \infty = \infty$ if $a > 0$; $a \cdot \infty = -\infty$ if $a < 0$;
- $a \div \infty = 0$ if $-\infty < a < \infty$;
- $\infty^a = \infty$ if $a > 0$; $\infty^a = 0$ if $a < 0$;
- $a^\infty = \infty$ if $a > 1$; $a^\infty = 0$ if $-1 < a < 1$;
- $\sqrt[a]{\infty} = \infty$ if $0 < a < \infty$;
- $\sqrt[\infty]{a} = 1$ if $0 < a < \infty$.

Finally, we can even divide by zero sometimes, *if* we are computing limits!

- $\lim_D (u/v) = \infty$ if $\lim_D u > 0$, $\lim_D v = 0$, and $v > 0$ in the direction D;
- $\lim_D (u/v) = -\infty$ if $\lim_D u > 0$, $\lim_D v = 0$, and $v < 0$ in the direction D;
- $\lim_D (u/v) = -\infty$ if $\lim_D u < 0$, $\lim_D v = 0$, and $v > 0$ in the direction D;
- $\lim_D (u/v) = \infty$ if $\lim_D u < 0$, $\lim_D v = 0$, and $v < 0$ in the direction D;
- $\lim_D (u/v)$ is neither ∞ nor $-\infty$ (although you can say that it is $\pm\infty$) if $\lim_D u \neq 0$, $\lim_D v = 0$, and neither $u/v > 0$ nor $u/v < 0$ is eventually true in the direction D.

In other words, if $v \to 0$ with a consistent sign, then the limit of u/v is plus or minus infinity, depending on how the sign of v compares to the sign of u, as long as u approaches something other than 0. However, this tells us nothing if $u \to 0$ too; for that, you must manipulate the expression algebraically or use a more advanced technique such as L'Hôpital's Rule (page 24) or expansion into power series (page 49).

3 Differentiation

The single most important topic in Calculus is probably *differentiation*. Whereas limits tell us *where* a quantity is going as it changes, differentiation tells us *how quickly* the quantity is changing. Technically, the question answered by limits does come up more often, but it's also trivial to solve in the vast majority of practical cases (when the variable is given by a continuous function); it may not seem that way while you're doing the problems, but that's just because we're focussing on the exceptions. Differentiation, however, is rarely trivial. That said, it is also rarely difficult; you just need to learn the rules.

A word about notation: As I remarked earlier (on page 3), when $y = f(x)$, we can write $\mathrm{d}y/\mathrm{d}x = f'(x)$; both sides of the latter equation are notation for a *derivative*, which is one of the things that differentiation produces. The left-hand side means the derivative of y with respect to x, while f' in the right-hand side is a function which is the derivative of the original function f. To say that the derivative of f is f' suggests that the derivative is a basic concept, not a combination of anything more complicated, and that is how I will approach derivatives at first. But the left-hand side suggests that a derivative is a ratio, the result of dividing $\mathrm{d}y$ by $\mathrm{d}x$, and this is also true. As for $\mathrm{d}y$ and $\mathrm{d}x$ themselves, they are the *differentials* of y and x; a differential is another thing that differentiation produces. Ultimately, it is more useful to think of a derivative in this way, but theoretically we start with derivatives of functions.

3.1 *Derivatives of functions*

Given any function f and a number c in the domain of f, the **difference quotient** of f at c is a function $\tilde{f}_c$, given by

$$\tilde{f}_c(h) = \frac{f(c+h) - f(c)}{h}.$$

Note that $\tilde{f}_c$ is not defined at 0. (In general, it's defined at any value h such that $h \neq 0$ and f is defined at $c+h$.) The **derivative** of f at c is the limit of $\tilde{f}_c$ approaching 0:

$$f'(c) = \lim_{h\to 0} \tilde{f}_c(h) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

(When this exists, we say that f is **differentiable** at c.) This is the definition in the textbook (see page 116), except that the book doesn't bother to give a name to $\tilde{f}_c$.

Because limits are closely related to continuity, it's possible to give a definition of the derivative based on continuity. First, extend the definition of $\tilde{f}_c$ like this:

$$\tilde{f}_c(h) = \begin{cases} \dfrac{f(c+h) - f(c)}{h} & \text{for } h \neq 0, \\ f'(c) & \text{for } h = 0. \end{cases}$$

If there exists a unique number $f'(c)$ that makes this function continuous at 0, then that number is the derivative of f at c; if there isn't, then this derivative doesn't exist and f is not differentiable at c. As it is, this is just the usual definition stated with different terminology. Now I'll do a little algebra on $\tilde{f}_c$: if $h \neq 0$ and f is defined at $c+h$, then

$$\begin{aligned} \tilde{f}_c(h) &= \frac{f(c+h) - f(c)}{h}, \\ h\,\tilde{f}_c(h) &= f(c+h) - f(c), \\ h\,\tilde{f}_c(h) + f(c) &= f(c+h), \\ f(c+h) &= f(c) + \tilde{f}_c(h)\,h; \end{aligned}$$

if $h = 0$, then this equation is still true as long as $\tilde{f}_c$ is defined at 0, since then it just says that $f(c) = f(c)$. So another way to define the derivative is to say that f is differentiable at c if there exists a function $\tilde{f}_c$ that is continuous at 0 and satisfies the last equation above (for all h such that f is defined at $c+h$), and then $f'(c) = \tilde{f}_c(0)$. One reason that this is useful is that having the entire function $\tilde{f}_c$ can help with proving theorems about derivatives; see the next section.

3.2 *Theorems about derivatives*

Every operation has a corresponding rule for derivatives. To begin with, recall that if f and g are functions, then $f+g$ is another function, which is defined wherever both f and g are defined, and whose values are given by $(f+g)(x) = f(x) + g(x)$. We similarly have $f-g$, fg and f/g (but the last of these is undefined wherever the value of g is zero, even if f and g are both defined there).

The theorems about their derivatives are as follows:

- The Sum Rule: $(f+g)' = f' + g'$,
- The Difference Rule: $(f-g)' = f' - g'$,
- The Product Rule: $(fg)' = f'g + fg'$,
- The Quotient Rule: $(f/g)' = \dfrac{f'g - fg'}{g^2}$. These are equations about functions; you can also put an argument into them:

$$\begin{aligned}(f+g)'(x) &= f'(x) + g'(x),\\ (f-g)'(x) &= f'(x) - g'(x),\\ (fg)'(x) &= f'(x)g(x) + f(x)g'(x);\\ (f/g)'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.\end{aligned}$$

A general strategy to prove these is to apply the equation for $f(c+h)$. For example, to prove that fg is differentiable wherever f and g are, with $(fg)' = f'g + fg'$, I'll use $\tilde{f}_c$ and $\tilde{g}_c$ along with the limit definition of $(fg)'$:

$$\begin{aligned}(fg)'(c) &= \lim_{h\to 0} \frac{(fg)(c+h) - (fg)(c)}{h} = \lim_{h\to 0} \frac{f(c+h)\,g(c+h) - f(c)\,g(c)}{h}\\
&= \lim_{h\to 0} \frac{\Big(f(c) + \tilde{f}_c(h)\,h\Big)\Big(g(c) + \tilde{g}_c(h)\,h\Big) - f(c)\,g(c)}{h}\\
&= \lim_{h\to 0} \frac{f(c)\,g(c) + f(c)\,\tilde{g}_c(h)\,h + \tilde{f}_c(h)\,h\,g(c) + \tilde{f}_c(h)\,h\,\tilde{g}_c(h)\,h - f(c)\,g(c)}{h}\\
&= \lim_{h\to 0} \frac{\tilde{f}_c(h)\,g(c)\,h + f(c)\,\tilde{g}_c(h)\,h + \tilde{f}_c(h)\,\tilde{g}_c(h)\,h^2}{h} = \lim_{h\to 0} \Big(\tilde{f}_c(h)\,g(c) + f(c)\,\tilde{g}_c(h) + \tilde{f}_c(h)\,\tilde{g}_c(h)\,h\Big)\\
&= \tilde{f}_c(0)\,g(c) + f(c)\,\tilde{g}_c(0) + \tilde{f}_c(0)\,\tilde{g}_c(0)\,0 = f'(c)\,g(c) + f(c)\,g'(c) + f'(c)\,g'(c)\,0\\
&= f'(c)\,g(c) + f(c)\,g'(c).\end{aligned}$$

(To evaluate the limit near the end, I need $\tilde{f}_c$ and $\tilde{g}_c$ to be continuous at 0.) I used smaller steps than the textbook does on page 133 (which is the only reason that my proof is longer), and I think that it's a little more straightforward, without the part where you add and subtract something without knowing yet why it will help.

The derivative of a constant function is the constant zero function; that is, if $f(x) = K$ for all x, where K is some constant, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{K - K}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

This fact may be called the Constant Rule. Using this, a special case of the Product Rule is the Multiple Rule:

$$(kf)'(x) = kf'(x)$$

if k is a constant. Another useful rule is the Power Rule: If $f(x) = x^n$ for all x, where n is a constant, then

$$f'(x) = nx^{n-1}.$$

(For integer values of n, this may be proved by repeated application of the Product and Quotient Rules, and there is a more complicated argument that applies to other rational values of n; however, a complete proof is easiest after considering exponents and logarithms.)

Using these rules, you can differentiate any polynomial function, or more generally any rational function. For a polynomial, you differentiate term by term (allowed by the Sum Rule), ignoring any constant terms (by the Constant Rule). For each term, you apply the Multiple Rule (to leave any coefficients alone) and the Power Rule (to bring down the exponent as a coefficient and subtract one from that exponent). For example, if $f(x) = 3x^4 - 5x^2 + 2x - 12$, then $f'(x) = 3(4x^{4-1}) - 5(2x^{2-1}) + 2(1x^{1-1}) + 0 = 12x^3 - 10x + 2$. For rational functions, you must also apply the Quotient Rule. There are examples in Section 3.3 of the textbook and in my video online.

3.3 *The Chain Rule*

One more rule, very important for theoretical purposes, is the Chain Rule. Using this, I'll be able to justify a new notation for derivatives and an even faster way to calculate them, so in the end you won't need to refer to the Chain Rule explicitly. However, we need it first to ensure that the new technique will work!

Here is the Chain Rule in function notation:

If g is differentiable at c and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c and

$$(f \circ g)'(c) = f'\big(g(c)\big)\, g'(c).$$

Here, $f \circ g$ is the *composite* of f after g, defined by $(f \circ g)(x) = f\big(g(x)\big)$.

I'll prove this using $\tilde{g}_c$ and $\tilde{f}_{g(c)}$:

$$\begin{aligned}
(f \circ g)'(c) &= \lim_{h\to 0} \frac{(f \circ g)(c+h) - (f \circ g)(c)}{h} = \lim_{h\to 0} \frac{f\big(g(c+h)\big) - f\big(g(c)\big)}{h} \\
&= \lim_{h\to 0} \frac{f\big(g(c) + \tilde{g}_c(h)\, h\big) - f\big(g(c)\big)}{h} = \lim_{h\to 0} \frac{f\big(g(c)\big) + \tilde{f}_{g(c)}\big(\tilde{g}_c(h)\, h\big)\, \tilde{g}_c(h)\, h - f\big(g(c)\big)}{h} \\
&= \lim_{h\to 0} \frac{\tilde{f}_{g(c)}\big(\tilde{g}_c(h)\, h\big)\, \tilde{g}_c(h)\, h}{h} = \lim_{h\to 0} \Big(\tilde{f}_{g(c)}\big(\tilde{g}_c(h)\, h\big)\, \tilde{g}_c(h)\Big) \\
&= \tilde{f}_{g(c)}\big(\tilde{g}_c(0)\, 0\big)\, \tilde{g}_c(0) = \tilde{f}_{g(c)}\big(g'(c)\, 0\big)\, g'(c) = \tilde{f}_{g(c)}(0)\, g'(c) \\
&= f'\big(g(c)\big)\, g'(c).
\end{aligned}$$

This proof is as straightforward as something so abstract can be, and it can be done immediately and rigorously without postponing things as the textbook does. I have the definition of derivative using $\tilde{f}_c$ to thank for this; this definition of derivative will be handy for some other proofs later on, such as for the Mean Value Theorem.

One immediately useful consequence of the Chain Rule is a generalized form of the Power Rule (what the textbook calls the Power Chain Rule): If g is differentiable at c and n is a constant, then g^n is also differentiable at c (where $(g^n)(x)$ is defined as $g(x)^n$), and $(g^n)'(c) = ng(c)^{n-1}g'(c)$. The reason is that g^n is a composite $f \circ g$ where f is the power function given by $f(x) = x^n$.

3.4 *Differentials*

Many calculations in calculus are easier to do using *differentials*. Furthermore, differentials and the related *differential forms* are often used in applications, especially (but not only) to physics. The official textbook covers differentials (in Section 3.11), but incompletely and only in one minor application. It then uses differentials again later (mostly in material for Calculus 2 and 3), but they are useful much earlier. So I will make heavy use of them.

If x is a variable quantity, then $\mathrm{d}x$ is the **differential** of x. You can think of $\mathrm{d}x$ as indicating an infinitely small (infinitesimal) change in the value of x, or (better) the amount by which x changes when an infinitesimal change is made (an infinitely small change in the value of the independent variable t). A precise definition is in the next section, but you will *not* be tested directly on that; what you need to know is how to *use* differentials.

Note that $\mathrm{d}x$ is *not* d times x, and $\mathrm{d}x$ is also *not* exactly a function of x. Rather, x (being a *variable* quantity) should itself be a function of some other quantity t, and $\mathrm{d}x$ is also a function of a sort; so d is an *operator*: something that turns one function into another function. However, an expression like $u\,\mathrm{d}x$ does involve multiplication: it is u times the differential of x.

We often divide one differential by another; for example, $\mathrm{d}y/\mathrm{d}x$ is the result of dividing the differential of y by the differential of x. The textbook introduces this notation early to stand for the *derivative* of y with respect to x, and indeed it is that; but what the book doesn't tell you is that $\mathrm{d}y/\mathrm{d}x$ literally is $\mathrm{d}y$ divided by $\mathrm{d}x$. Unfortunately, $\mathrm{d}^2y/\mathrm{d}x^2$, the second derivative of y with respect to x, is *not* literally $\mathrm{d}^2y = \mathrm{d}(\mathrm{d}y)$ divided by $\mathrm{d}x^2 = (\mathrm{d}x)^2$; for this reason, I prefer the notation $(\mathrm{d}/\mathrm{d}x)^2y$, meaning $(\mathrm{d}/\mathrm{d}x)(\mathrm{d}/\mathrm{d}x)y = (\mathrm{d}/\mathrm{d}x)(\mathrm{d}y/\mathrm{d}x) = \mathrm{d}(\mathrm{d}y/\mathrm{d}x)/\mathrm{d}x$ for the second derivative.

The most important fact about differentials is this: If f is a differentiable function, then

$$\mathrm{d}\Big(f(u)\Big) = f'(u)\,\mathrm{d}u.$$

That is, the differential of $f(u)$ equals $f'(u)$ times the differential of u, where f' is the derivative of the function f. This fact not only shows the relationship between differentials and derivatives, but also (because u could be any quantity) it encapsulates the **Chain Rule** in differential form. The Chain Rule is an important principle in calculus, which is often difficult to learn how to use; but with differentials it is easy.

For example, suppose that you have discovered (say from the definition as a limit) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. Then this fact can be expressed in differential form:

$$\mathrm{d}(x^2) = 2x\,\mathrm{d}x. \qquad (*)$$

Conversely, if (by performing a calculation with differentials) you discover the equation (*) above, then you know the derivative of f as well:

$$f'(x) = \frac{\mathrm{d}\Big(f(x)\Big)}{\mathrm{d}x} = \frac{\mathrm{d}(x^2)}{\mathrm{d}x} = \frac{2x\,\mathrm{d}x}{\mathrm{d}x} = 2x.$$

Whichever of these facts you discover first, once you know them, you know something even more general:

$$\mathrm{d}(u^2) = 2u\,\mathrm{d}u.$$

(The power to derive this from equation (*) is the Chain Rule.) The value of this is that u can be any expression whatsoever; for example, if $u = x^2$ again, then

$$\mathrm{d}(x^4) = \mathrm{d}\Big((x^2)^2\Big) = 2(x^2)\,\mathrm{d}(x^2) = 2x^2(2x\,\mathrm{d}x) = 4x^3\,\mathrm{d}x.$$

So now you have learnt a new derivative, without having to calculate it from scratch.

Every theorem about derivatives of functions may also be expressed as a theorem about differentials. Here are the most common rules:

- The Constant Rule: $\mathrm{d}(K) = 0$ if K is constant.
- The Sum Rule: $\mathrm{d}(u + v) = \mathrm{d}u + \mathrm{d}v$.
- The Translate Rule: $\mathrm{d}(u + C) = \mathrm{d}u$ if C is constant.
- The Difference Rule: $\mathrm{d}(u - v) = \mathrm{d}u - \mathrm{d}v$.
- The Product Rule: $\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v$.
- The Multiple Rule: $\mathrm{d}(ku) = k\,\mathrm{d}u$ if k is constant.
- The Quotient Rule: $\mathrm{d}\left(\dfrac{u}{v}\right) = \dfrac{v\,\mathrm{d}u - u\,\mathrm{d}v}{v^2}$.
- The Power Rule: $\mathrm{d}(u^n) = nu^{n-1}\,\mathrm{d}u$ if n is constant.
- The Root Rule: $\mathrm{d}(\sqrt[m]{u}) = \dfrac{\sqrt[m]{u}\,\mathrm{d}u}{mu}$ if m is constant.

Of these, only the Constant Rule, the Sum Rule, the Product Rule, and the Power Rule are absolutely necessary, since every other expression built out of the operations in the rules above can be built out of the operations in these four rules. However, it is often handy to use all of these rules; it is up to you how many of these rules to learn. (The Power Rule given here really corresponds to the *Generalized* Power Rule in the textbook, because it incorporates the Chain Rule within it. The Root Rule is not in the textbook, because a root can be algebraically transformed into a power; but the version here rationalizes the denominator, which can be convenient.)

In addition, every time that you learn the derivative of a new function, you learn a new rule for differentials, by applying the Chain Rule to that function. I already showed you an example of this on page 16: applying the Chain Rule to the function $f(x) = x^2$ gives the special case of the Power Rule for $n = 2$. Here are a few other functions whose derivatives you will learn, expressed as rules for differentials:

- $\mathrm{d}(\exp u) = \exp u\,\mathrm{d}u$.
- $\mathrm{d}(\ln u) = \dfrac{\mathrm{d}u}{u}$.
- $\mathrm{d}(\sin u) = \cos u\,\mathrm{d}u$.
- $\mathrm{d}(\cos u) = -\sin u\,\mathrm{d}u$.
- $\mathrm{d}(\operatorname{atan} u) = \dfrac{\mathrm{d}u}{u^2 + 1}$.

And more! (To be clear, $\exp u$ means e^u, $\ln u = \log_{\mathrm{e}} u$, u is in radians in $\sin u$ and $\cos u$, and $\operatorname{atan} u$ is what is also written $\arctan u$, $\operatorname{Tan}^{-1} u$, or $\tan^{-1} u$ and gives a result in radians.)

Notice that every one of these rules turns the differential on the left into a sum of terms (possibly only one term, or none in the case of the Constant Rule), each of which is an ordinary expression multiplied by a differential (or something algebraically equivalent to this). An expression like this is called a **differential form** (although actually there are more general sorts of differential forms). If, when you are calculating the differential of an expression, your result at any stage is *not* like this, then you have made a mistake!

3.5 *Defining differentials*

To formally define what differentials are and prove their properties, I'll make the same assumption that I made at the beginning of these notes, that there is an independent variable t that every other variable is a function of. Then, I said that if $u = f(t)$, then $u|_{t=c} = f(c)$. Now I'll say that, if $u = f(t)$ and the function f is differentiable, then

$$\mathrm{d}u\big|_{\substack{t=c,\\ \mathrm{d}t=h}} = f'(c)\,h.$$

More generally, if $u = f(t)$ and $v = g(t)$, then

$$(u\,\mathrm{d}v)\big|_{\substack{t=c,\\ \mathrm{d}t=h}} = f(c)\,g'(c)\,h.$$

Again, this is abstract, but the concrete application is straightforward; for example:

$$(2x\,\mathrm{d}x + 3\,\mathrm{d}x)\big|_{\substack{x=4,\\ \mathrm{d}x=0.05}} = 2(4)(0.05) + 3(0.05) = 0.55,$$
$$(2x\,\mathrm{d}x + 3y\,\mathrm{d}y)\big|_{\substack{x=4,y=5,\\ \mathrm{d}x=0.05,\mathrm{d}y=0.02}} = 2(4)(0.05) + 3(5)(0.02) = 0.7.$$

(I've put small numbers in for $\mathrm{d}x$ and $\mathrm{d}y$, because this is most often what comes up in practice, although for theoretical purposes it doesn't matter.) It's now more common to be given only partial information; for example:

$$(2x\,\mathrm{d}x + 3\,\mathrm{d}x)\big|_{x=4} = 2(4)\,\mathrm{d}x + 3\,\mathrm{d}x = 11\,\mathrm{d}x,$$
$$(2x\,\mathrm{d}x + 3y\,\mathrm{d}y)\big|_{\substack{x=4,\\ y=5}} = 2(4)\,\mathrm{d}x + 3(5)\,\mathrm{d}y = 8\,\mathrm{d}x + 15\,\mathrm{d}y.$$

Notice that you *don't* plug in the values of x and y inside the differential operator d; if you're not given values of $\mathrm{d}x$ and $\mathrm{d}y$, then those differentials must remain in the answer.

While expressions like the above come up occasionally (see the discussion of linear approximation on pages 21 and 22), the main purpose of a precise definition is to prove theorems. (That's how we can be sure that the rules of Calculus will always work, at least when the definitions that prove them can be made to apply.) Earlier I gave a list of rules for differentials; we can prove these using the precise definition of differential and the known rules for derivatives of functions. For example, if $u = f(t)$ and $v = g(t)$, then $uv = f(t)\,g(t) = (fg)(t)$. Therefore,

$$\mathrm{d}(uv)\big|_{\substack{t=c,\\ \mathrm{d}t=h}} = (fg)'(c)\,h = \Big(f'(c)\,g(c) + f(c)\,g'(c)\Big)\,h = g(c)\,f'(c)\,h + f(c)\,g'(c)\,h = (v\,\mathrm{d}u + u\,\mathrm{d}v)\big|_{\substack{t=c,\\ \mathrm{d}t=h}}.$$

Here, I've used the formal definition of differential along with the Product Rule for derivatives of functions. The conclusion is that $\mathrm{d}(uv)$ and $v\,\mathrm{d}u + u\,\mathrm{d}v$ always evaluate to the same result, so

$$\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v,$$

which is the Product Rule for differentials. In the same way, all of the rules for differentials follow from rules for derivatives of functions.

The Chain Rule is an important special case, so I'll prove it too. If $u = g(t)$ and f is any function, then $f(u) = f(g(t)) = (f \circ g)(t)$, so if f is differentiable, then

$$\mathrm{d}\Big(f(u)\Big)\Big|_{\substack{t=c,\\ \mathrm{d}t=h}} = \mathrm{d}\Big((f \circ g)(t)\Big)\Big|_{\substack{t=c,\\ \mathrm{d}t=h}} = (f \circ g)'(c)\,h = f'\Big(g(c)\Big)\,g'(c)\,h = \Big(f'(u)\,\mathrm{d}u\Big)\Big|_{\substack{t=c,\\ \mathrm{d}t=h}}.$$

Again, I used the definition of differential and the Chain Rule for functions, and my conclusion is the Chain Rule for differentials:

$$\mathrm{d}\Big(f(u)\Big) = f'(u)\,\mathrm{d}u$$

whenever f is a differentiable function.

It's not really essential to assume that there exists a *single* independent variable that every other variable is a function of, and we'll stop making that assumption in Calculus 3 (if you stick around that long). Then the formal definition will become a little trickier, but all of the rules for differentials will continue to apply exactly as I stated them above.

3.6 *Using differentials*

The main technique for using differentials is simply to take the differential of both sides of an equation. However, you may only do this to an equation that holds *generally*, but *not* to an equation that holds only for *particular* values of the variables. (Ultimately, this is because d is an operator, not a function, so it must be applied to entire functions, not only to particular values of those functions.)

The simplest case is an equation such as $y = \exp(x^2)$, when we want the derivative of y with respect to x. So:

$$\begin{aligned} y &= \exp(x^2); \\ \mathrm{d}y &= \mathrm{d}(\exp(x^2)) = \exp(x^2)\,\mathrm{d}(x^2) = \exp(x^2)\cdot 2x\,\mathrm{d}x = 2x\exp(x^2)\,\mathrm{d}x; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\exp(x^2). \end{aligned}$$

Now we have the derivative. If we want the second derivative, then we do this again:

$$\begin{aligned} \mathrm{d}y/\mathrm{d}x &= 2x\exp(x^2); \\ \mathrm{d}(\mathrm{d}y/\mathrm{d}x) &= \mathrm{d}\Big(2x\exp(x^2)\Big) = \exp(x^2)\,\mathrm{d}(2x) + 2x\,\mathrm{d}\Big(\exp(x^2)\Big) \\ &= \exp(x^2)\cdot 2\,\mathrm{d}x + 2x\cdot 2x\exp(x^2)\,\mathrm{d}x = \big(2\exp(x^2) + 4x^2\exp(x^2)\big)\,\mathrm{d}x; \\ (\mathrm{d}/\mathrm{d}x)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} &= 2\exp(x^2) + 4x^2\exp(x^2). \end{aligned}$$

Now we have the second derivative (also written $\mathrm{d}^2y/\mathrm{d}x^2$).

The previous example began with an equation solved for y. But we don't need this; suppose instead that we have $y^5 + x^2 = x^5 + y$ (which *cannot* be solved for either variable using the usual algebraic operations of addition, subtraction, multiplication, division, powers, and roots). Undaunted, we forge ahead anyway:

$$\begin{aligned} y^5 + x^2 &= x^5 + y; \\ \mathrm{d}(y^5 + x^2) &= \mathrm{d}(x^5 + y); \\ \mathrm{d}(y^5) + \mathrm{d}(x^2) &= \mathrm{d}(x^5) + \mathrm{d}y; \\ 5y^{5-1}\,\mathrm{d}y + 2x^{2-1}\,\mathrm{d}x &= 5x^{5-1}\,\mathrm{d}x + \mathrm{d}y; \\ 5y^4\,\mathrm{d}y - \mathrm{d}y &= 5x^4\,\mathrm{d}x - 2x\,\mathrm{d}x; \\ (5y^4 - 1)\,\mathrm{d}y &= (5x^4 - 2x)\,\mathrm{d}x; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{5x^4 - 2x}{5y^4 - 1}. \end{aligned}$$

This process is called **implicit differentiation**.

The second derivative is a little more straightforward at first (or it would be if we didn't have to use the Quotient Rule), but there is a twist at the end:

$$\begin{aligned} \mathrm{d}y/\mathrm{d}x &= \frac{5x^4 - 2x}{5y^4 - 1}; \\ \mathrm{d}(\mathrm{d}y/\mathrm{d}x) &= \mathrm{d}\left(\frac{5x^4 - 2x}{5y^4 - 1}\right) = \frac{(5y^4 - 1)\,\mathrm{d}(5x^4 - 2x) - (5x^4 - 2x)\,\mathrm{d}(5y^4 - 1)}{(5y^4 - 1)^2} \\ &= \frac{(5y^4 - 1)(20x^3 - 2)\,\mathrm{d}x - (5x^4 - 2x)(20y^3)\,\mathrm{d}y}{(5y^4 - 1)^2} \\ &= \frac{20x^3 - 2}{5y^4 - 1}\,\mathrm{d}x - \frac{20y^3(5x^4 - 2x)}{(5y^4 - 1)^2}\,\mathrm{d}y; \\ (\mathrm{d}/\mathrm{d}x)^2 y = \frac{\mathrm{d}(\mathrm{d}y/\mathrm{d}x)}{\mathrm{d}x} &= \frac{20x^3 - 2}{5y^4 - 1} - \frac{20y^3(5x^4 - 2x)}{(5y^4 - 1)^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{20x^3 - 2}{5y^4 - 1} - \frac{20y^3(5x^4 - 2x)}{(5y^4 - 1)^2}\,\frac{5x^4 - 2x}{5y^4 - 1} \end{aligned}$$

(which could be simplified further). Notice that I substitute the known expression for $\mathrm{d}y/\mathrm{d}x$ in the last step.

Another handy application of differentials is the case where both quantities x and y may be expressed as functions of some other quantity t. (For the purposes of formal definitions, we always assume that this is possible, but now we're really going to use it.) If we start with the same equation as above, then this will give us an equation relating the derivatives with respect to t:

$$\begin{aligned} y^5 + x^2 &= x^5 + y; \\ \mathrm{d}(y^5 + x^2) &= \mathrm{d}(x^5 + y); \\ \mathrm{d}(y^5) + \mathrm{d}(x^2) &= \mathrm{d}(x^5) + \mathrm{d}y; \\ 5y^{5-1}\,\mathrm{d}y + 2x^{2-1}\,\mathrm{d}x &= 5x^{5-1}\,\mathrm{d}x + \mathrm{d}y; \\ 5y^4\frac{\mathrm{d}y}{\mathrm{d}t} + 2x\frac{\mathrm{d}x}{\mathrm{d}t} &= 5x^4\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}. \end{aligned}$$

If we have information about one or both of these derivatives, then this equation will often give us useful information to solve a problem. This situation is called **related rates**, since derivatives can be viewed as rates of change (especially derivatives with respect to time t, although the t in the equation above doesn't have to stand for time).

When we get to integrals, differentials become so useful that even the textbook starts using them, but I'll save that for later.

3.7 *Derivatives with respect to time*

Derivatives with respect to time are a major application of Calculus. Here are some examples:

Quantity:	Derivative (with respect to time):	Second derivative:	Third derivative:
Position	Velocity	Acceleration	Jerk
Velocity	Acceleration	Jerk	
Speed	Colloquial acceleration		
Acceleration	Jerk		
Net wealth	Net income		
National debt	National deficit		

Position tells you where something is, while **velocity** tells you how it is moving, that is how its position is changing with time. Velocity is not quite the same thing as **speed**, since velocity keeps track of direction as well. (In this class, most problems involving motion will take place in only one dimension, so there are two directions, represented by positive and negative velocity, while speed is the absolute value of velocity.)

The derivative of velocity with respect to time, in other words the second derivative of position with respect to time, is **acceleration** in the technical sense of this term. On the other hand, the derivative of speed is **colloquial acceleration**, which reflects how the term is used in everyday life. Colloquially, we say that an object is accelerating if its speed increases with time (in other words if it is speeding up) and decelerating if its speed decreases (in other words if it is slowing down). But in the technical sense of the term, if an object is moving in the negative direction and slows down, then its velocity is becoming less negative and more positive, and so its acceleration is positive, even though its colloquial acceleration is negative. (For motion in more than one dimension, its even possible for the colloquial acceleration to be zero even while the technical acceleration is far from zero; this happens when changing direction while travelling at a constant speed.)

The time derivative of acceleration (in the technical sense) is **jerk**; that makes jerk the second derivative of velocity and the third derivative of position. Whereas position and velocity can't be directly felt, you feel acceleration as a pressure or absence thereof (a sense of falling or being held or pushed), and a sudden change in that acceleration is a jerk or yank. In engineering, acceleration must be controlled because it can destroy objects by crushing; jerk must be controlled because it can destroy objects by breaking them apart. Even higher derivatives are sometimes also studied, although the terminology varies.

Turning to finances, your **net wealth** is the total value of all assets that you own minus the value of all of your debts. (If you owe more than you own, then your net wealth is negative.) This is measured in units of money, such as dollars. Your **net income**, on the other hand, is the total value of everything that you receive (as wages, gifts, and so forth) in a period of time minus the value of your expenses. This is measured in units of money per unit of time, such as dollars per year. In finance, the default unit of time is a year, so you'll often say that someone's income is so many dollars, but this really means so many dollars *per year*. Unlike physical motion, money goes in and comes out in discrete chunks, so the continuous ideas of Calculus are only an approximation, but they can be a good approximation for some purposes.

Turning from personal finances to national, a country's government usually has some debt, called the country's **national debt**, and if the government spends more than it receives from taxes and other revenue, then the difference is the **national deficit**. The debt is the total amount of money owed by the government, while the deficit is the additional amount that has to be borrowed in a given period of time. Again, deficit should really be measured in units of money per unit of time; so if someone says the the U.S. national deficit is nearly 500 billion dollars, this really means 500 billion dollars *per year*. This is the same as 5000 billion dollars (or 5 trillion dollars) per decade (since a decade is 10 years). On the other hand, when they say that the U.S. national debt is nearly 20 trillion dollars, then they are saying exactly what they mean; this is the net result of all of the deficits (and occasional surpluses, which are negative deficits) in the past.

In 2010, there was a widely cited economics paper (Reinhart & Rogoff) that argued that a country tended towards economic disaster as its government's debt approached its GDP (gross domestic product, a measure of a country's overall income). In 2013, a review (Herndon, Ash, & Pollin) found statistical errors that undemined the paper's conclusions, and this made the mainstream news media for a while. This should have just been the normal process of science: a flawed idea being corrected. But it was big news because Reinhart & Rogoff had struck an intuitive chord; it made sense that of course your debt should always be well below your ability to pay it off. But in fact that only sounds reasonable if you ignore the units! Reinhart & Rogoff's conclusion was really that a country was courting disaster if its government's debt was close to its GDP *times one year*; otherwise, the units of measurement don't make sense. The idea that a country should have enough income to pay off its government's debt becomes the idea that a country should have enough income to pay off its government's debt *in one year* (if all income were devoted to this purpose), and there's no intuitive reason why that should be necessary to avoid economic ruin. (It is still true that a country's economy tends to be better off when its government debt divided by its GDP is lower than otherwise, and it's conceivable that there could be some reason that there's something special about when that quotient is close to one year; but there isn't.)

3.8 *Linear approximation*

Recall from page 13 above that if f is differentiable at c, then

$$f(c+h) = f(c) + \tilde{f}_c(h)\,h$$

for some function $\tilde{f}_c$ that's continuous at 0 (and then $\tilde{f}_c(0)$ is $f'(c)$). Since $\tilde{f}_c$ is continuous at 0, we can say that $\tilde{f}_c(h) \approx \tilde{f}_c(0)$ when $h \approx 0$, or in other words, $\tilde{f}_c(h) \approx f'(c)$ when $h \approx 0$. Putting this approximation in the equation above, we get

$$f(c+h) \approx f(c) + f'(c)\,h$$

when $h \approx 0$. Writing x for $c+h$ (so that $h = x-c$), you can also put this as

$$f(x) \approx f(c) + f'(c)\,(x-c)$$

when $x \approx c$. While the left-hand side could be any differentiable function, the right-hand side is a linear function of x; this function is the **linear approximation** to f near c, or the **linearization** of f near c.

The textbook likes to name this function L; so $f(x) \approx L(x) = f(c) + f'(c)\,(x-c)$. I don't like that name, because which function you get as the linear approximation depends on which function you start with as well as on which number c you look at. So I write $L_{f,c}$ for the linearization of f near c:

$$f(x) \approx L_{f,c}(x) = f(c) + f'(c)\,(x-c).$$

This is actually only the beginning of a whole sequence of approximations, each (typically) better than the one before it:

$f(x) \approx f(c)$, a constant, if f is continuous at c;

$f(x) \approx f(c) + f'(c)\,(x-c)$, a linear function of x, if f is differentiable at c;

$f(x) \approx f(c) + f'(c)\,(x-c) + \dfrac{1}{2}\,f''(c)\,(x-c)^2$, a quadratic function of x, if f is twice differentiable at c;

$f(x) \approx f(c) + f'(c)\,(x-c) + \dfrac{1}{2}\,f''(c)\,(x-c)^2 + \dfrac{1}{6}\,f'''(c)\,(x-c)^3$, a cubic function of x,

if f is 3-times differentiable at c;

$\vdots$

(This sequence of approximations is covered in Calculus 2; see Section 9.8 of the textbook and page 45 of these notes.)

It's handy to describe linear approximation in terms of differentials and differences. While a differential represents an infinitesimal (infinitely small) change, a **difference** represents an appreciable or finitesimal (meaning *not* infinitely small) change. As x changes from c to $c+h$, we say that the difference in x is

$$\Delta x = (c+h) - c = h.$$

Meanwhile, if $y = f(x)$, then the difference in y is

$$\Delta y = y|_{x=c+h} - y|_{x=c} = f(c+h) - f(c).$$

To be specific, we can write

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} = f(c+h) - f(c).$$

Then the linear approximation says that

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} = f(c+h) - f(c) \approx f(c) + f'(c)\,h - f(c) = f'(c)\,h = \mathrm{d}y|_{\substack{x=c,\\ \mathrm{d}x=h}}.$$

So in the end, the linear approximation replaces differences with differentials. Although

$$\Delta y|_{\substack{x=c,\\ \Delta x=h}} \approx \mathrm{d}y|_{\substack{x=c,\\ \mathrm{d}x=h}}$$

is the proper way to put it, often one abbreviates this as

$$\Delta y \approx \mathrm{d}y.$$

(But really this only correct if we also have $\Delta x = \mathrm{d}x$, or at least $\Delta x \approx \mathrm{d}x$, because that difference is also replaced by a differential in the approximation.)

More generally, you can say that an equation involving differentials can be replaced by an approximate equation involving differences. For example, if $x^5 + 2x = y^5 + y$, then $5x^4\,\mathrm{d}x + 2\,\mathrm{d}x = 5y^4\,\mathrm{d}y + \mathrm{d}y$ (by differentiating both sides), so $5x^4\,\Delta x + 2\,\Delta x \approx 5y^4\,\Delta y + \Delta y$. Then if you are looking near the only obvious solution, $(x, y) = (0, 0)$, and you want to know the value of y when $x = 0.3$ (so $\Delta x = 0.3 - 0 = 0.3$, you find $5(0)^4(0.3) + 2(0.3) \approx 5(0)^4\,\Delta y + \Delta y$, so $\Delta y \approx 0.6$; in other words, the new y-value is approximately $0 + 0.6 = 0.6$. (The actual solution to $(0.3)^5 + 2(0.3) = y^5 + y$ is $y|_{x=0.3} \approx 0.55$ to 2 decimal places, but I couldn't do that by hand!)

It can be important to know how far off an approximation might be, and this is basically given by the next term in the sequence of approximations on the top of the page. To be specific, the Mean-Value Theorem (see pages 23 and 24) says that $f(x) - f(c)$ (which is the error in the constant approximation $f(x) \approx f(c)$) cannot be any larger in absolute value than $|x - c|$ times the maximum value that f' takes between x and c; similarly, $f(x) - L_{f,c}(x)$ (which is the error in the linear approximation near c) cannot be any larger in absolute value than $|x - c|^2$ times half the maximum value that f'' takes between x and c. However, the details of why this is so are best saved for the full treatment of the entire sequence of approximations that begins on page 45 of these notes.

3.9 *Newton's Method*

If you want to solve an equation $f(x) = 0$, then the Intermediate Value Theorem may give you a way to approximate the solution, but it is usually very inefficient. The Newton–Raphson Method (or simply Newton's Method) is usually much faster, although it doesn't always work. Here, you start with a guess x_0, then replace it with a (hopefully) better guess x_1, and so on. These guesses are computed in turn as follows:

$$x_1 = x_0 + \frac{f(x_0)}{f'(x_0)},$$
$$x_2 = x_1 + \frac{f(x_1)}{f'(x_1)},$$
$$x_3 = x_2 + \frac{f(x_2)}{f'(x_2)},$$
$$\vdots$$

With any luck, none of these guesses will give $f'(x) = 0$ (which makes the next guess undefined) but eventually one will give $f(x) \approx 0$ to as close an approximation as one wants.

The Newton–Raphson Method is guaranteed to work under certain conditions given by the Newton–Kantorovich Theorem: If f is differentiable at a, $f(a)$ and $f'(a)$ are nonzero, f is twice differentiable strictly between a and $a - 2f(a)/f'(a)$, and

$$|f''(x)| \leq \frac{|f'(a)|^2}{2\,|f(a)|}$$

whenever x is strictly between a and $a - 2f(a)/f'(a)$, then Newton's Method will give a sequence of values that are strictly between a and $a - 2f(a)/f'(a)$, and that converge to a solution of $f(x) = 0$ in the sense that the limit $\lim_{n\to\infty} x_n$ exists and $f(\lim_{n\to\infty} x_n) = 0$.

3.10 *Advanced theorems*

There are various theorems about derivatives and differentials that should seem obvious if you understand the basic idea, but mathematicians have still proved them just to be safe.

For example, the derivative of a function is supposed to tell us how much the output is changing relative to the input. In particular, if the derivative is positive, then the output should increase when the input increases and decrease when the input decreases; conversely, if the derivative is negative, then the output should decrease when the input increases and decrease when the input increases. The first kind of function is called *increasing* and the other is *decreasing*; there are precise theorems that a function whose derivative somewhere is positive or negative must be increasing or decreasing (repsectively) near there. Conversely, if a function has a local extremum, then the derivative must be either zero or undefined there. This fact is key to optimization (see page 25 and following).

Another group of theorems are the mean-value theorems. The point of a derivative is that it can be approximated by a difference quotient; the mean-value theorems reverse this, and show how a difference quotient must (under some conditions) be equal to a derivative somewhere nearby. All of these theorems consider a function f defined on at least an interval $[a, b]$ (with $a < b$) such that f is continuous on all of $[a, b]$ and differentiable at least between a and b (but possibly not at a or b themselves).

Specifically, Rolle's mean-value theorem says

> If $f(b) - f(a) = 0$, then $f'(c) = 0$ for some c between a and b.

Then Lagrange's mean-value theorem says

> In any case, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c between a and b.

Finally, Cauchy's mean-value theorem says

If g is another function satisfying the same conditions as f and if furthermore g' is never zero between a and b, then $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$ for some c between a and b.

In Cauchy's mean-value theorem, I like to think of $f(x)$ as u and $g(x)$ as v, so that the left-hand side is $\mathrm{d}u/\mathrm{d}v$ (evaluated at $x = c$) while the right-hand side is $\Delta u/\Delta v$ (evaluated at $x = a$ and $\Delta x = b - a$). Lagrange's theorem is the special case of Cauchy's theorem where $g(x)$ is always simply x, and Rolle's theorem is the special case of Lagrange's theorem where $f(b) - f(a) = 0$.

3.11 *L'Hôpital's Rule*

One important consequence of Cauchy's mean-value theorem is **L'Hôpital's Rule**. This is a rule for limits again, but it handles limts with forms such as $\infty \div \infty$ and $0 \div 0$.

L'Hôpital's Rule applies when taking limits in any direction D, if u and v are two quantities defined in the direction D, so long as either $\lim_D (1/v) = 0$ (so $\lim_D v = \pm\infty$ in other words) or both $\lim_D u$ and $\lim_D v$ are zero. In that case, if $\lim_D (\mathrm{d}u/\mathrm{d}v)$ exists, then $\lim_D (u/v)$ also exists and the two limits are equal.

L'Hôpital's Rule can also be applied to limits with exponents by taking logarithms, applying the rule directly, and reversing the logarithms. It is therefore very versatile, although Taylor series (see page 49) can do even more.

3.12 *Concavity*

There are various terms used when the values of a function, its average rates of change, or its second average rates of change (the rates of change of the rates of change) are all positive (or negative), at least on some interval. When the function is differentiable, and especially when it's twice differentiable, there are easier ways to describe these. This is all summarized in the table below.

Property of f:	Definition:	If differentiable:	If twice differentiable:
Positive	$f(a) > 0$	—	—
Negative	$f(a) < 0$	—	—
Increasing	$\dfrac{f(b) - f(a)}{b - a} > 0$	$f'(a) > 0$	—
Decreasing	$\dfrac{f(b) - f(a)}{b - a} < 0$	$f'(a) < 0$	—
Concave upward	$\dfrac{\dfrac{f(c) - f(b)}{c - b} - \dfrac{f(b) - f(a)}{b - a}}{c - a}$	$\dfrac{f'(b) - f'(a)}{b - a} > 0$	$f''(a) > 0$
Concave downward	$\dfrac{\dfrac{f(c) - f(b)}{c - b} - \dfrac{f(b) - f(a)}{b - a}}{c - a}$	$\dfrac{f'(b) - f'(a)}{b - a} < 0$	$f''(a) < 0$

In all of these, the function f has the given property on some interval if the given condition holds whenever a, b, and c are *distinct* numbers in that interval. (They must be distinct to avoid division by zero.)

Generally, it's much easier to work with the rightmost condition for every property, but you can't do that if the necessary derivatives don't exist. Even if the function isn't differentiable at all, it still makes sense to say whether or not it's concave upward or downward.

Incidentally, here is some other terminology that you may see for these properties:

- Sometimes people use $\geq$ and $\leq$ in place of $>$ and $<$. If you want to be clear, you can use adverbs: 'strictly' for the definitions above (using $>$ and $<$) or 'weakly' for the versions with $\geq$ and $\leq$.
- Sometimes people put the word 'monotone' in front of 'increasing' and 'decreasing', even though it really isn't necessary. (However, when people use this word, they are more likely to mean ‹weakly› too, even if they don't say so.)
- Alternatively, if the word 'monotone' is used alone, then it means ‹increasing› (probably ‹weakly increasing›); the corresponding word for ‹decreasing› (usually ‹weakly decreasing›) is 'antitone' (but this word is fairly rare).

- If the word 'concave' is used alone, then it means ‹concave downard›; the corresponding word for ‹concave upward› is 'convex' (and this word is extremely common). Again, people who use this terminology are more likely to mean ‹weakly›.

3.13 *Graphing*

If you want to have a complete graph of a function f, then these are all of the things that you should make sure show up:

- $x = 0$, if f is defined at that point;
- $x \to -\infty$, if f is defined in that direction;
- $x \to \infty$, if f is defined in that direction;
- $x \to c^-$, if f is defined in that direction, whenever f is undefined or discontinuous at c;
- $x \to c^+$, if f is defined in that direction, whenever f is undefined or discontinuous at c;
- $x = c$, if f is defined at that point, whenever f is undefined approaching c from either direction (or both);
- $x = c$, whenever $f(c) = 0$;
- $x = c$, whenever f' is undefined or discontinuous at c, if f is defined there;
- $x = c$, whenever $f'(c) = 0$;
- $x = c$, whenever f'' is undefined or discontinuous at c, if f is defined there;
- $x = c$, whenever $f''(c) = 0$.

This should be sufficient whenever f is a twice-differentiable function whose domain is an interval, or more generally whenever f is *piecewise twice-differentiable*: a piecewise-defined function in which the domain of each piece is an interval and in which each piece is twice-differentiable except possibly at its endpoints. (There are weirder functions that can't be put in this form, but you shouldn't have to deal with them in this class.)

If you have a graphing calculator, then you may use it, but you still need to ensure that all of the features listed above appear. At the very least, this may require you to adjust the calculator's graphing window. If you're graphing by hand, then you'll get the best results if you know the values or limits of f, f', and f'' for all of these, but you should at least get f for all of them and f' whenever you looked there because of something involving f' or f''. You can also look at points in between these (assuming that f is defined there).

3.14 *Optimization*

Literally, **optimization** is making something the best, but we use it in math to mean **maximization**, which is making something the biggest. (You can imagine that the thing that you're maximizing is a numerical measure of how good the thing that you're optimizing is.) Essentially the same principles apply to **minimization**, which is making something the smallest. (And *pessimization* is making something the worst, although people don't use that term very much.) A generic term for making something the largest or smallest is **extremization**.

In theory, optimization is simply finding absolute extrema, which is most easily done for continuous functions on closed, bounded intervals. In that case, the maximum and minimum must both exist, by the Extreme Value Theorem, and each of them must occur at either the endpoint of the interval or where the derivative of the function is either zero or undefined. However, practical problems cannot always be modelled in this way, so we will need some more general techniques.

The key principle of applied optimization is this:

> A quantity u can only take a maximum or minimum value when its differential $\mathrm{d}u$ is zero or undefined.

If you write u as $f(x)$, where f is a fixed differentiable function and x is a quantity whose range of possible values you already understand (typically an interval), then $\mathrm{d}u = f'(x)\,\mathrm{d}x$. So u can only take an extreme value when its derivative (with respect to x) is zero or undefined or when you can no longer vary x however you please (which must occur at the extreme values of x and typically only then). This recreates the situation that I referred to above, finding the extreme values of a function defined on an interval. However, the principle that $\mathrm{d}u$ is zero or undefined applies even when u is not explicitly given as a function of anything else.

Be careful, because u might not have a maximum or minimum value! Assuming that u varies continuously (which it must if Calclulus is to be useful at all), then it must have a maximum and minimum value whenever the range of possibilities is *compact*; this means that if you pass continuously through the possibilities in any way, then you are always approaching some limiting possibility. (In terms of $u = f(x)$, this is the case when f is continuous and its domain, the range of possible values of x, is a closed and bounded interval.)

However, if the range of possibilities heads off to infinity in some way, or if there is an edge case that's not quite possible to reach, then you also have to take a limit to see what value u is approaching. (In terms of $u = f(x)$, if the interval is open or unbounded at either end, then there is a direction in which x could vary but in which there is no limiting value of x in the range of possibilities.) If any such limit is larger than every value that u actually reaches (which includes the possibility that a limit is ∞), then u has no maximum value; if any such limit is smaller than every value that u actually reaches (which includes the possibility that a limit is $-\infty$), then u has no minimum value.

So in the end, you look at these possibilities:

- when the derivative of u is zero or undefined,
- the extreme edge cases, and
- the limits approaching impossible limiting cases.

The largest value of u that you find in this way (regardless of whether this value is actually attained or is only approached in the limit) is called the *supremum* of u; similarly, the smallest value of u that you find is called the *infimum* of u. If u actually takes the value of its supremum, then that same value is also the *maximum* of u; but if u only approaches its supremum in a limit, then it has no maximum. Similarly, if u actually takes the value of its infimum, then that same value is also the *minimum* of u; but if u only approaches its infimum in a limit, then it has no minimum.

Here is a typical problem: The hypotenuse of a right triangle (maybe it's a ladder leaning against a wall) is fixed at 20 feet, but the other two sides of the triangle could be anything. Still, since it's a right triangle, we know that $x^2 + y^2 = 20^2$, where x and y are the lengths of legs of the triangle. Differentiating this, $2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$. Now suppose that we want to maximize or minimize the area of this triangle. Since it's a right triangle, the area is $A = \frac{1}{2}xy$, so $\mathrm{d}A = \frac{1}{2}y\,\mathrm{d}x + \frac{1}{2}x\,\mathrm{d}y$. If this is zero, then $\frac{1}{2}y\,\mathrm{d}x + \frac{1}{2}x\,\mathrm{d}y = 0$, to go along with the other equation $2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$.

The equations at this point will always be linear in the differentials, so think of this is a system of linear equations in the variables $\mathrm{d}x$ and $\mathrm{d}y$. There are various methods for solving systems of linear equations; I'll use the method of addition (aka elimination), but any other method should work just as well. So $\frac{1}{2}y\,\mathrm{d}x + \frac{1}{2}x\,\mathrm{d}y = 0$ becomes $2xy\,\mathrm{d}x + 2x^2\,\mathrm{d}y = 0$ (multiplying both sides by $4x$), while $2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$ becomes $2xy\,\mathrm{d}x + 2y^2\,\mathrm{d}y = 0$ (multiplying both sides by y). Subtracting these equations gives $(2x^2 - 2y^2)\,\mathrm{d}y = 0$, so either $\mathrm{d}y = 0$ or $x^2 = y^2$. Now, x and y can change freely as long as they're positive, but we have limiting cases: $x \to 0^+$ and $y \to 0^+$. Since $x^2 + y^2 = 400$, we see that $x^2 \to 400$ as $y \to 0$; since x is positive, this means that $x \to 20$ as $y \to 0$. Similarly, $y \to 20$ as $x \to 0$. In those cases, $A = \frac{1}{2}xy \to 0$. On the other hand, if $x^2 = y^2$, then $x = y$ (since they are both positive), so $x, y = 10\sqrt{2}$, since $x^2 + y^2 = 400$. In that case, $A = \frac{1}{2}xy = 100$.

So the largest area is 100 square feet, and while there is no smallest area, the area can get arbitrarily small with a limit of 0.

3.15 *Economic applications*

In word problems in economics or finance, a few quantities arise regularly, which you should know about.

- **Quantity** in this context has a specific meaning: the amount of a good or service made and/or sold in a given period of time. Quantity is thus measured in such units as pounds per week, items per year, or litres per hour. Quantity is variously denoted q or x.
- **Price** (or *unit price*) is the amount of money received for a given amount of goods or services. So price is measured in units such as dollars per pound or euros per item. Price is denoted p, a *lowercase* letter.
- **Revenue** is the amount of money received for goods or services in a given period of time. Revenue is measured in dollars per week, euros per year, etc. Revenue is denoted R, and we have this equation:
$$R = qp.$$
(Notice that the units make sense in this equation; amount over time, multiplied by money over amount, becomes money over time.)
- **Cost** is the amount of money that the business has to spend (in a given period of time) in order to produce and distribute their goods and services. (In this terminology, *cost* is completely different from *price*.) Like revenue, cost is measured in units of money over time.
- Finally, **profit** is the amount of money that the business makes and keeps in a given period of time. Unlike everything else here, it makes sense for profit to be negative. Profit is denoted P, an *uppercase* letter, and we have another equation:
$$P = R - C.$$

In business, you generally want to maximize profit: make it not only positive but as large as possible. Even if you don't want to maximize profit as normally measured (because you care about something else besides money), economists typically try to calculate whatever else you care about and still say that you maximize profit (in a generalized sense).

For any of these quantities, we can discuss their average or marginal values. In this context, the **average** profit/cost/etc is the profit/cost/etc divided by the quantity:
$$\bar{P} = \frac{P}{q},\ \bar{C} = \frac{C}{q},\ \ldots.$$
(As you can see, a bar is used to indicate this ratio. Be careful; when we get to applications of integrals, this bar will be used to denote an average in a different way.) On the other hand, the **marginal** profit/cost/etc is the derivative of profit/cost/etc with respect to quantity:
$$P' = \frac{\mathrm{d}P}{\mathrm{d}q},\ C' = \frac{\mathrm{d}C}{\mathrm{d}q},\ \ldots.$$
(As you can see, a prime tick is used to indicate this derivative, which is safe in context because it always means the derivative respect to q. For a derivative with respect to time, which is also important in this context even though we aren't doing any examples of that in this class, a dot may be used instead.) Although the units for a marginal or average quantity are the same, they represent different things!

Finally, people also speak of the **marginal average** profit/cost/etc:
$$\bar{P}' = \frac{\mathrm{d}(P/q)}{\mathrm{d}q} = \frac{qP' - P}{q^2} = \overline{P' - \bar{P}},$$
$$\bar{C}' = \frac{\mathrm{d}(C/q)}{\mathrm{d}q} = \frac{qC' - C}{q^2} = \overline{C' - \bar{C}},$$
$$\vdots$$

The marginal profit is particularly important, since it must be zero when profit is maximized (as long as the maximum profit occurs when it is still possible to vary the quantity in any way desired); and since the marginal marginal profit (the second derivative of profit with respect to quantity) is typically negative, the profit really will be maximized when the marginal profit is zero. However, in the absence of information about the revenue, there is a rule of thumb that one should minimize the average cost instead, which means finding where the marginal average cost is zero.

4 Integrals

This is a summary of the concepts of integral calculus.

4.1 *Definite integrals*

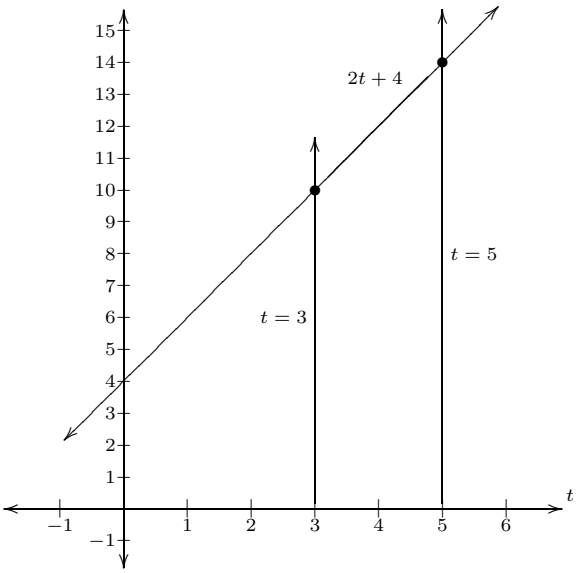
Just as the *differential* of a finite quantity is an infinitesimal (infinitely small) change in that quantity, so the **definite integral** of an infinitesimal quantity is the sum of infinitely many values of that quantity, giving a finite result. If x and y are standard quantities (neither infinitely large nor infintely small), then $y\,\mathrm{d}x$ is a typical infinitesimal quantity. (An expression like this is called a *differential form*.) If we add this up from the point where $x = a$ to the point where $x = b$, then we get the **definite integral**

$$\int_{x=a}^{b} y\,\mathrm{d}x.$$

As long as the same variable x is used throughout, then it's safe to abbreviate this as

$$\int_{a}^{b} y\,\mathrm{d}x.$$

For example, $\int_3^5 (2t+4)\,\mathrm{d}t$ is the sum, as t varies smoothly from 3 to 5, of the product of $2t+4$ and $\mathrm{d}t$ (the infinitesimal change in t) at each stage along the way. We can think of this product as giving the area of a rectangle whose height is $2t+4$ and whose width is $\mathrm{d}t$; if we line these rectangles up side by side, then they combine to give a trapezoid:



We can find out the area of this trapezoid using geometry, since its width is $5 - 3 = 2$ and its height varies linearly from $2(3) + 4 = 10$ to $2(5) + 4 = 14$. Therefore,

$$\int_3^5 (2t+4)\,\mathrm{d}t = \frac{10+14}{2} \cdot 2 = 24.$$

Normally, you can't evaluate an integral by drawing a picture like this; I'll come back to how we can calculate it after a brief digression.

4.2 *Antidifferentials*

If $\mathrm{d}u = y\,\mathrm{d}x$, then $y\,\mathrm{d}x$ is the *differential* of u, as you know. We also say that u is an **antidifferential** of $y\,\mathrm{d}x$. However, u is *not the only* antidifferential of $y\,\mathrm{d}x$; if C is any constant, then $\mathrm{d}(u + C) = y\,\mathrm{d}x$ too, so $u + C$ is also an antidifferential of $y\,\mathrm{d}x$. However, for a continuously defined quantity, there is *no other* antidifferential of $y\,\mathrm{d}x$. Even if there are gaps in the definition of the quantity, we can say that $u + C$ is an antidifferential of $\mathrm{d}u$ if and only if C is a *local* constant, meaning that it can change value only across a gap where u is undefined. (Ultimately, this is a consequence of the theorem that if the derivative of a function on an interval is always zero, then that function must be a constant; the relevant function here is the difference between the functions that give any two possible antidifferentials.)

Antidifferentials are denoted by '$\int$', so we have

$$\int \mathrm{d}u = u + C$$

by definition. (This looks similar to the notation for a definite integral, which makes sense reasons that will be explained below, but you can tell the difference because there are no bounds attached to the symbol.) For example,

$$\mathrm{d}(t^2 + 4t) = 2t\,\mathrm{d}t + 4\,\mathrm{d}t = (2t + 4)\,\mathrm{d}t,$$

so

$$\int (2t + 4)\,\mathrm{d}t = \int \mathrm{d}(t^2 + 4t) = t^2 + 4t + C.$$

As $2t + 4$ is the derivative of $t^2 + 4t$ with respect to t, we also say that $t^2 + 4t$ is an **antiderivative** of $2t + 4$ with respect to t. An antidifferential or antiderivative is also called an **indefinite integral**; so 'indefinite integral of $(t^2 + 4)\,\mathrm{d}t$' (antidifferential) and 'indefinite integral of $t^2 + 4$ with respect to t' (antiderivative) both mean $\int (t^2 + 4)\,\mathrm{d}t$.

To find antidifferentials (or antiderivatives), we must run the rules for differentials (and derivatives) backwards. This is often a subtle process, which I'll return to after a brief digression.

4.3 *The Fundamental Theorem of Calculus*

The **Fundamental Theorem of Calculus** relates definite and indefinite integrals. There are two parts:

1. $\mathrm{d}\left(\int_{t=a}^{b} f(t)\,\mathrm{d}t\right) = f(b)\,\mathrm{d}b - f(a)\,\mathrm{d}a$;
2. $\int_{t=a}^{b} \mathrm{d}f(t) = f(b) - f(a)$.

The first part applies whenever f is a continuous function (assuming that a and b are differentiable quantities); in particular, it claims that the integral exists and is differentiable. The second part applies whenever f is a differentiable function (assuming that t is a differentiable quantity); in particular, it claims that the integral exists.

Although both of these parts refer directly to definite integrals, indefinite integrals (antidifferentials) appear implicitly because of the presence of the differentials. Specifically, the first part claims that the definite integral that appears in it is an antidifferential of the differential form on its right-hand side, and the second part shows how to evaluate a definite integral of a differential form whose antidifferential is known.

If you want to express these without refering to the function f, then you can write them thus:

1. $\mathrm{d}\left(\int_{a}^{b} \omega\right) = \omega|_{a}^{b}$;
2. $\int_{a}^{b} \mathrm{d}u = u|_{a}^{b}$.

Here, I'm using ω to stand for an entire differential form (for which people often use Greek letters) and $u|_{a}^{b}$ is short for $u|_{b} - u|_{a}$. These basically say that d and $\int$ cancel as long as you move the bounds on the integral into bounds on a difference.

It's the second part of the theorem that we use the most. If you want to evaluate a definite integral $\int_{a}^{b} y\,\mathrm{d}x$, then you should first figure out the indefinite integral $\int y\,\mathrm{d}x$. If the answer to this is u (or more generally $u + C$), then this means that $y\,\mathrm{d}x = \mathrm{d}u$; that is, u is an antidifferential of $y\,\mathrm{d}x$. Therefore, $\int_{x=a}^{b} y\,\mathrm{d}x =$

$\int_{x=a}^{b} \mathrm{d}u$, and the FTC tells us that this is equal to $u|_{x=a}^{b}$. As this last expression is simply a difference, you can figure it out using simple algebra.

For example, consider

$$\int_{t=3}^{5} (2t+4)\,\mathrm{d}t.$$

In the last section, we saw that $\int (2t+4)\,\mathrm{d}t = t^2 + 4t + C$; in other words, $(2t+4)\,\mathrm{d}t = \mathrm{d}(t^2+4t)$. Therefore,

$$\begin{aligned}\int_3^5 (2t+4)\,\mathrm{d}t &= \int_3^5 \mathrm{d}(t^2+4t) = (t^2+4t)|_3^5 \\ &= \Big((5)^2 + 4(5)\Big) - \Big((3)^2 + 4(3)\Big) = (45) - (21) = 24.\end{aligned}$$

(Notice that this is the same answer as when I did this using geometry!)

This also explains why the same term 'integral' and symbol '$\int$' are used for both the definite integral (a sum of infinitely small quantities) and the indefinite integral (the antidifferential). They at first appear to be completely different concepts, but in reality they are closely related, through the Fundamental Theorem of Calculus.

4.4 *Integration techniques*

This leaves us with one problem: how do we find indefinite integrals?

Each rule for differentiation gives us a rule for integration. In the table below, I have some rules for differentiation (all of which you should know by now), together with corresponding rules for integration:

$$\begin{aligned}
\mathrm{d}(u+v) &= \mathrm{d}u + \mathrm{d}v, & \int (y+z)\,\mathrm{d}x &= \int y\,\mathrm{d}x + \int z\,\mathrm{d}x; \\
\mathrm{d}(ku) &= k\,\mathrm{d}u \text{ (when } k \text{ is constant)}, & \int ky\,\mathrm{d}x &= k\int y\,\mathrm{d}x \text{ (when } k \text{ is constant)}; \\
\mathrm{d}(uv) &= v\,\mathrm{d}u + u\,\mathrm{d}v, & \int u\,\mathrm{d}v &= uv - \int v\,\mathrm{d}u; \\
\mathrm{d}(u^n) &= nu^{n-1}\,\mathrm{d}u \text{ (when } n \text{ is constant)}, & \int u^m\,\mathrm{d}u &= \frac{1}{m+1}u^{m+1} + C \text{ (when } m \neq -1 \text{ is constant)}; \\
\mathrm{d}(\mathrm{e}^u) &= \mathrm{e}^u\,\mathrm{d}u, & \int \mathrm{e}^u\,\mathrm{d}u &= \mathrm{e}^u + C; \\
\mathrm{d}(\ln|u|) &= \frac{1}{u}\,\mathrm{d}u, & \int \frac{1}{u}\,\mathrm{d}u &= \ln|u| + C; \\
\mathrm{d}(\sin u) &= \cos u\,\mathrm{d}u, & \int \cos u\,\mathrm{d}u &= \sin u + C; \\
\mathrm{d}(\cos u) &= -\sin u\,\mathrm{d}u, & \int \sin u\,\mathrm{d}u &= -\cos u + C; \\
&\text{etc.}
\end{aligned}$$

Using these rules, you can work out all of the integrals in the textbook through Chapter 6, and then some.

For example, to find $\int (2t+4)\,\mathrm{d}t$:

$$\int (2t+4)\,\mathrm{d}t = \int 2t\,\mathrm{d}t + \int 4\,\mathrm{d}t = 2\int t^1\,\mathrm{d}t + 4\int \mathrm{d}t = 2\Big(\frac{1}{2}t^2\Big) + 4t + C = t^2 + 4t + C.$$

This is the same answer as we got before, but this time I didn't have to guess the answer and get lucky; I was able to actually calculate it. That's how you're going to be doing most of the problems.

For more complicated integrals, there are fancier techniques. Rather than learn all of these, you can program them into a computer. There are even free websites that will do this for you!

4.5 *Summary*

To find the indefinite integral $\int y\,\mathrm{d}x$, you need to use integration techniques; your answer will still have the variable in it and should end with a new local-constant term C. To find the definite integral $\int_a^b y\,\mathrm{d}x$, first find the indefinite integral and then take a difference; assuming a and b are constants, your answer will also be constant (and the C will disappear).

So for example, to find the definite integral of $2t+4$ with respect to t from 3 to 5:

$$\int_3^5 (2t+4)\,\mathrm{d}t = \int_3^5 (2t^1\,\mathrm{d}t + 4\,\mathrm{d}t) = \left(2\left(\frac{1}{2}t^2\right) + 4t\right)\Big|_3^5 = (t^2+4t)|_3^5 = 45-21 = 24.$$

This is simply a combination of calculations that I did earlier, to find the indefinite integral and to apply the FTC.

4.6 *Semidefinite integrals*

Besides the *definite* integral $\int_a^b f(x)\,\mathrm{d}x$ and the *indefinite* integral $\int f(x)\,\mathrm{d}x$, there is also a **semidefinite integral** $\int_a f(x)\,\mathrm{d}x$. While the definite integral works out to a specific value (as long as f, a, and b are specified), the indefinite and semidefinite integrals still have the variable x in them. On the other hand, while the indefinite integral depends on an arbitrary C, the definite and semidefinite integrals don't have this. So the semidefinite integral fits in between the other two kinds.

Here is one way to define it:

$$\int_{x=a} f(x)\,\mathrm{d}x = \int_{t=a}^{x} f(t)\,\mathrm{d}t.$$

That is, introduce a new variable t and use the old variable x as the upper bound of a definite integal. The Second Fundamental Theorem of Calculus,

$$\int_{x=a}^{b} f(x)\,\mathrm{d}x = \Big(\int f(x)\,\mathrm{d}x\Big)\Big|_{x=a}^{b} = \Big(\int f(x)\,\mathrm{d}x\Big)\Big|_{x=b} - \Big(\int f(x)\,\mathrm{d}x\Big)\Big|_{x=a},$$

also tells us how to evaluate semidefinite integrals:

$$\int_{x=a} f(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x - \Big(\int f(x)\,\mathrm{d}x\Big)\Big|_{x=a}.$$

In other words, work out the indefinite integral as usual; then, instead of evaluating this at two values of the variable before subtracting, evalute it at one value and keep the variable in the other expression (then subtract). For example,

$$\int_{x=1} x\,\mathrm{d}x = \frac{x^2}{2} - \left(\frac{x^2}{2}\right)\Big|_{x=1} = \frac{x^2}{2} - \left(\frac{(1)^2}{2}\right) = \frac{1}{2}x^2 - \frac{1}{2}.$$

(You can probably skip the step with $|_{x=1}$ in it, since once you've written down $x^2/2$ before the minus sign, you can immediately plug in 1 for x to get $(1)^2/2$ after the minus sign.)

4.7 *Integration by parts*

Integration by parts is based on the Product Rule for differentiation. In terms of differentials, the Product Rule says that $\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v$. Taking indefinite integrals of both sides and rearranging the terms slightly, this becomes

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Unlike integration by substitution, you don't rewrite the problem in terms of u (nor v). Instead, you identify suitable u and v and their differentials and then write out the equation above in terms of x (or whatever your variable is).

You want to pick u and v so that $\int u\,\mathrm{d}v$ is the integral that you care about, which means splitting up the factors of the integrand, some into u and some into $\mathrm{d}v$. Once you know u and $\mathrm{d}v$, you can find $\mathrm{d}u$ and v, at least if you know how to integrate whatever $\mathrm{d}v$ is. (When you do this integration of $\mathrm{d}v$ to get v, you have a choice up to a local constant; you're deciding what v is, so just pick the simplest expression.) If you split things up well, then $\int v\,\mathrm{d}u$ will be simpler than what you started with.

Here is my advice on how to split factors into u and $\mathrm{d}v$ so that integration by parts will make the next integral easier. The items on the top of the list are the best choices for $\mathrm{d}v$, and the items on the bottom are the best choices for u. Put as many factors as you can into $\mathrm{d}v$, starting at the top of this list and working your way to the bottom, as long as you still have something that you know how to integrate to get v. Then put whatever factors are left over into u.

- $\mathrm{d}x$ (this *must* go into $\mathrm{d}v$),
- $\mathrm{e}^x$and other exponential expressions,
- $\sin x$ and other trigonometric expressions,
- polynomials and other algebraic expressions,
- $\ln x$ and other logarithmic expressions,
- $\operatorname{asin} x = \sin^{-1} x$ and other inverse trigonometric expressions.

In complicated cases, you may have to use integration by parts more than once. Just keep going until either you get something that you can handle or you get back to where you started. In the latter case, you can set up an equation to solve for your integral.

5 Differential equations

A **differential equation** is an equation with differentials or derivatives in it. Here are three examples of differential equations:

$$\begin{aligned} f'(x) &= 3f(x); \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 3y; \\ \mathrm{d}y &= 3y\,\mathrm{d}x. \end{aligned}$$

In fact, these three examples are all basically equivalent. If you are given the first of these, then you should make up a name for $f(x)$, say y, and turn the first equation into the middle one. And in the middle equation, you should clear fractions to turn it into the last one. (But any of these might be the original form, depending on how the equation is thought up in the first place.)

To actually solve this equation, you can use the technique of **separation of variables**. After reaching the last equation, notice that x only appears on the right-hand side but y appears on both sides. If you divide both sides by y, however, then y appears only on the left-hand side. (If $y = 0$, then dividing by y is invalid; I'll come back to that later.) Then the variables are separated:

$$\frac{\mathrm{d}y}{y} = 3\,\mathrm{d}x.$$

(If you're ever unsure which side to put which variable on, then try to put the differentials in the numerators of any fractions. In this example, $1/\mathrm{d}x = 3y/\mathrm{d}y$ would have the variables separated just as much, but it would be less useful, because the next step, below, wouldn't work.)

Now take the indefinite integral of each side of the equation:

$$\begin{aligned} \int \frac{\mathrm{d}y}{y} &= \int 3\,\mathrm{d}x; \\ \ln|y| + C_1 &= 3x + C_2; \\ \ln|y| &= 3x + C_2 - C_1. \end{aligned}$$

Each integral gives an arbitrary constant, and I subtracted to put them both on the right-hand side. However, since $C_2 - C_1$ could itself be any constant, you can just write this as

$$\ln|y| = 3x + C.$$

In practice, you can skip the other steps with constants and just remember to tack a constant onto the last integral in the equation.

We're not finished; this equation is no longer a differential equation, but it also hasn't been solved for anything. If we want to solve it for y, then we still need to do some algebra to get y by itself on its side of the equation:

$$\begin{aligned} |y| &= \mathrm{e}^{3x+C}; \\ y &= \pm\mathrm{e}^{3x+C}. \end{aligned}$$

(If you're given an equation in x and y, then it's a good bet that they want you to solve for y; if you're given an equation like the first example with a function in it, then it's a good bet that they want you to solve for the function. But in principle, you could solve any of these equations for x instead.)

There is one mistake here, which is the step where I divided by y. If $y = 0$, then this is invalid. Furthermore, if $y = 0$ always, then the equation is true, because then both sides of the original equation (in any of the three forms) are 0. (This sort of special exception is fairly common with differential equations.) So a complete solution is

$$y = \pm\mathrm{e}^{3x+C} \text{ or } y = 0.$$

You can make the final solution look a bit nicer by writing $\pm e^{3x+C}$ as $\pm e^C e^{3x}$ and then making up a name for $\pm e^C$, say P. Since e^C could be any positive number, P could be any positive or negative number; the exception $y = 0$ is captured by $P = 0$. So the nicest form of the final solution is

$$y = Pe^{3x},$$

where P is an arbitrary constant. (However, you shouldn't always expect to be able to do a simplifying trick like that.)

Of course, if the original form of the equation is the first example, then you should write this solution as

$$f(x) = Pe^{3x}.$$

5.1 *Initial-value problems*

An **initial-value problem** consists of a differential equation together with enough data to determine the arbitrary constants. Here are three examples of initial-value problems:

$$f'(x) = 3f(x),\ f(0) = 5;$$
$$\frac{dy}{dx} = 3y,\ y|_{x=0} = 5;$$
$$dy = 3y\,dx,\ y|_{x=0} = 5.$$

Again, these three examples are all basically equivalent; if $y = f(x)$, then $y|_{x=0}$ means $f(0)$.

There are two ways to solve an initial-value problem. One is to ignore the initial value and just solve the differential equation, at first. In this example, that gives us

$$y = Pe^{3x},$$

as you've seen. Then you put in the given values, which in this case gives

$$5 = Pe^{3(0)}.$$

Now you can solve for P:

$$5 = P(1);$$
$$P = 5.$$

Therefore, the final answer to the initial-value problem is

$$y = 5e^{3x}.$$

(Again, if the original form of the equation is the first example, then you should write this solution as $f(x) = 5e^{3x}$.)

Another technique is to solve the entire problem at once with the help of *semidefinite integrals* (page 31). Let's solve the example

$$dy = 3y\,dx,\ y|_{x=0} = 5$$

using semidefinite integrals. Again, separate the variables:

$$\frac{dy}{y} = 3\,dx.$$

Now instead of taking *indefinite* integrals of both sides, take *semidefinite* integrals, using the initial value to guarantee that you're doing the same thing to each side even though it's being done using different variables. In this case, since $y = 5$ when $x = 0$, a semidefinite integral starting at $y = 5$ is the same operation as a semidefinite integral starting at $x = 0$, so

$$\int_{y=5} \frac{dy}{y} = \int_{x=0} 3\,dx.$$

Evaluating these using the FTC gives

$$\ln|y| - \ln|5| = 3x - 3(0).$$

So compared to the integration without the initial value, the difference is that we know which specific constants to use in each integral. Now again, solve for y to finish:

$$\begin{aligned} \ln|y| &= 3x - 0 + \ln 5; \\ |y| &= \mathrm{e}^{3x+\ln 5}; \\ y &= \pm 5\mathrm{e}^{3x}. \end{aligned}$$

This is not completely perfect, because of the $\pm$, but we can figure this out by checking whether y really is 5 when $x = 0$; this will only be true if the sign is $+$. Finally, since we did again divide by y while solving this, check to make sure that y is never zero in the solution; it's not, so the final answer is

$$y = 5\mathrm{e}^{3x}.$$

Of course, this is the same solution as we got before, but this time we got the entire solution all at once without having to first get a solution with an arbitrary constant and then solving for the constant. You may solve intial-value problems using whichever method you prefer.

5.2 *Integrals as solutions to equations*

Although we normally solve a differential equation by taking integrals, you can also think of an integral as a solution to a differential equation. For example, the indefinite integral $\int f(x)\,\mathrm{d}x$ is the solution to the differential equation $\mathrm{d}y/\mathrm{d}x = f(x)$, and the semidefinite integral $\int_{x=a} f(x)\,\mathrm{d}x$ is the solution to the initial-value problem $(\mathrm{d}y/\mathrm{d}x = f(x), y|_{x=a} = 0)$. More generally, the solution to the initial-value problem $(\mathrm{d}y/\mathrm{d}x = f(x), y|_{x=a} = c)$ is $\int_{x=a} f(x)\,\mathrm{d}x + c$. These kinds of initial-value problems are in Sections 4.8 and 5.5 of the textbook and are covered in Calculus 1; more general differential equations and initial-value problems are in Section 7.2 and are covered in Calculus 2.

(There are even more general differential equations than I have discussed here, ones in which it is impossible the separate the variables in the equation; some of these are covered in Chapters 16 and 17 of the online-only version of the textbook. Yet more general differential equations are covered in SCC's course on differential equations, which is basically Calculus 4, but using a different textbook dedicated to that subject. Beyond that, there are graduate-level courses that you could take at a university; in fact, the study of differential equations is a major field of active research in mathematics. We are very far from knowing how to solve them all!)

A **sequence** is a function whose domain consists only of integers. (It's not necessary that all integers belong to the domain, just that nothing else does.) To emphasize that we're considering a sequence, people often write f_n instead of $f(n)$ when f is a sequence (and n is an integer in its domain). In fact, 'f' is not a very common name for a sequence; 'a' and 'x' (or letters near them) are much more common. Similarly, the argument of a sequence is usually denoted by a letter near the middle of the alphabet (usually between 'i' and 'n'), since these letters are often used for integers. (Still, as with any other variable, you can use any letter that you like in principle.) There is also some redundant terminology: instead of speaking of the input (or argument) and output (or value) of a function, we speak of an **index** and **term** of a sequence. For example, if $a_n = (-2)^n$, then the term with index 3 is $a_3 = (-2)^3 = -8$. (Sometimes people say that 8 is the 3rd term, but this really only works if a_n is undefined when $n < 1$.)

Since Calculus is about continuously varying quantities and a sequence has only discrete values (at most one for each integer), there's not much Calculus to be done with a sequence. Nevertheless, there is some: you can consider the limit of a sqeuence approaching infinity (or negative infinity). That is, while $\lim_{n\to c} a_n$ (for finite c), $\dfrac{\mathrm{d}a_n}{\mathrm{d}n}$, and $\int a_n \,\mathrm{d}n$ don't make sense, nevertheless $\lim_{n\to\infty} a_n$ and $\lim_{n\to-\infty} a_n$ can make sense. I'll focus on the first of these, which you can call simply the **limit** of the sequence, because many of our sequences will only be defined at natural numbers; however, limits approaching negative infinity really aren't much different.

Sometimes it's convenient to think of a sequence as the restriction to integers of some more general function. For example, if you're working with the sequence $a_n = 3n^2$, then you can think of the function $f(x) = 3x^2$; while f is defined for all real numbers and a is defined only for integers, otherwise they are the same thing. Since $\lim_{x\to\infty} f(x) = \infty$, this tells us that $\lim_{n\to\infty} a_n = \infty$ too. So most of the time, you can work out the limit of a sequence in the same way that you work out any other limit approaching infinity. If $a_n = f(n)$ for n an integer and f has a limit (possibly infinite) approaching infinity, then a has the same limit; this is a theorem. However, it's possible that a has a limit even when f does not, for example if $f(x) = \sin(\pi x)$. This has no limit as $x \to \infty$, since all values between -1 and 1 are taken for arbitrarily large values of x. When n is an integer, however, $\sin(\pi n) = 0$, so the limit of the sequence $a_n = \sin(\pi n)$ (which is really just the sequence $a_n = 0$) is 0.

There are some more systematic ways of turning a sequence into a function that's defined everywhere (or almost everywhere). These involve the floor and ceiling operations: the **floor** $\lfloor x \rfloor$ of a real number x is the largest integer that's not larger than x, and the **ceiling** $\lceil x \rceil$ of x is the smallest integer that's not smaller than x. Ever since you first learnt to round numbers up and down, you've been using these operations, even if you didn't have names for them; for example, $\lfloor 2.37 \rfloor = 2$ (round down to the nearest integer), and $\lceil 2.37 \rceil = 3$ (round up to the nearest integer). Be careful with negative numbers: $\lfloor -2.37 \rfloor = -3$, and $\lceil -2.37 \rceil = -2$. An important inequality about floors and ceilings is

$$\lfloor x \rfloor \leq x \leq \lceil x \rceil.$$

As long as x is itself fractional (that is not an integer), then

$$\lfloor x \rfloor < x < \lceil x \rceil,$$

and in that case you also have

$$\lfloor x \rfloor + 1 = \lceil x \rceil.$$

(But integers are an exception; if x is an integer, then $\lfloor x \rfloor$, x, and $\lceil x \rceil$ are all equal to each other.)

Using these operations, we can convert any sequence into a function defined more generally: if a is a sequence, then we can consider $a_{\lfloor x \rfloor}$ and $a_{\lceil x \rceil}$. If a is defined for all integers, then these will be defined for all real values of x; even if a isn't defined for all integers, still $a_{\lfloor x \rfloor}$ and $a_{\lceil x \rceil}$ will be defined for many more real numbers. And now we have this theorem:

$$\lim_{x\to\infty} a_{\lfloor x \rfloor} = \lim_{n\to\infty} a_n = \lim_{x\to\infty} a_{\lceil x \rceil}.$$

These functions $a_{\lfloor x \rfloor}$ and $a_{\lceil x \rceil}$ are unusual, since they are (for most sequences) discontinuous at every integer, but they can be handy to think about.

You can see a picture of these in Figure 9.11 on page 501 of the textbook. (The textbook is using this picture for a different purpose, although it is related, as you'll see later on.) In this picture, the book begins with a function f and then constructs a sequence a out of it by defining $a_n = f(n)$. Then on the top (Figure 9.11.a), it shows the graph of $y = f(x)$ in blue along with a graph of $y = a_{\lfloor x \rfloor} = f(\lfloor x \rfloor)$ in magenta; while on the bottom (Figure 9.11.b), it shows a graph of $y = f(x)$ in blue again but now with a graph of $y = a_{\lceil x \rceil} = f(\lceil x \rceil)$ in magenta. You'll notice that the sequence and all three of the other functions tend to the same limit (which in this case is 0). Even if the textbook had started with a function f that did not converge to a limit, the sequence and the two functions defined by floor and ceiling would still all converge to the same thing.

6.1 *Series*

I wrote above that you can't do much Calculus on sequences; in particular, I remarked that the derivative $\dfrac{\mathrm{d}a_n}{\mathrm{d}n}$ and integral $\int a_n \,\mathrm{d}n$ don't make sense. Ultimately, this is because $\mathrm{d}n$, an infinitesimal (infinitely small) but non-zero change in n, doesn't make sense when n takes only integer values; the smallest possible non-zero change in n is a change by 1, which is not infinitely small.

But there is something *analogous* to derivatives and integrals. The analogue to derivatives is the **difference** $\Delta_n a_n = a_{n+1} - a_n$, which is the difference of a_n with respect to n. (For example, $\Delta_n(3n) = 3(n+1) - 3n = 3$, and $\Delta_m(m^2) = (m+1)^2 - m^2 = 2m + 1$, which means that if $n = m^2$, then $\Delta_m n = 2\sqrt{n} + 1$.) Whereas the derivative is defined as a limit of difference quotients, the difference simply *is* a difference quotient where the change in n is $\Delta_n n = 1$. (Unfortunately, sequences do not have an analogue of the differential that will take care of changing from one variable to another. This is because $\Delta_u n \cdot \Delta_m u$ bears no particular relationship with $\Delta_m n$, even assuming that all of the values of u are integers.)

The analogue to an integral is a **series**, which is the result of adding up some of the terms of a sequence. (This word can be confusing, in two ways. The first is a quirk of grammar: the plural of 'series' is just 'series' again. You can say 'serieses' as the plural, although this is nonstandard, but using 'serie' as the singular is just plain wrong. The other confusing thing is that, in ordinary language, 'sequence' and 'series' mean basically the same thing; but in mathematics, a sequence is the more basic concept, being essentially just a list of numbers or other quantities, while a series is a sum that you build out of a sequence.)

Like differences, a finite series has no Calculus in it; you just add up some numbers. For example,

$$\begin{aligned}\sum_{n=3}^{7} (n^2+1) &= \Big((3)^2+1\Big) + \Big((4)^2+1\Big) + \Big((5)^2+1\Big) + \Big((6)^2+1\Big) + \Big((7)^2+1\Big) \\ &= 10 + 17 + 26 + 37 + 50 = 140.\end{aligned}$$

This means the sum of all of the values of $n^2 + 1$ as n runs from 3 to 7, taking only integer values along the way. That is, it's the sum of all of the values of $n^2 + 1$ as n takes the values 3, 4, 5, 6, and 7, which is what I calculated.

Strictly speaking, this is analogous to a proper integral such as $\int_{x=3}^{8} (x^2+1)\,\mathrm{d}x$. Actually, this is more than just an analogy: a series *is* an integral, albeit one whose Calculus content is trivial. Specifically,

$$\sum_{n=i}^{j} a_n = \int_{x=i}^{j+1} a_{\lfloor x \rfloor}\,\mathrm{d}x = \int_{x=i-1}^{j} a_{\lceil x \rceil}\,\mathrm{d}x.$$

(So in this example, $\sum_{n=3}^{7}(n^2+1) = \int_{x=3}^{8}\Big(\lfloor x \rfloor^2 + 1\Big)\,\mathrm{d}x$.) Since these are integrals of piecewise-constant functions, working them out is easy and just results in the original sum. So you don't want to evaluate a series by turning it into an integral; still, it can be handy to know that this can be done, becuase we know a lot of theorems about integrals that now automatically apply to series.

We traditionally speak of a sum from $i = a$ to $i = b$, written $\sum_{i=a}^{i=b}$ or simply $\sum_{i=a}^{b}$, where $b - a$ is a whole number $(0, 1, 2, \ldots)$; assuming for simplicity that a is an integer (so that b is also), this sum covers every integer i that satisfies the inequality $a \le i \le b$, or in other words all of the integers in the interval $[a, b]$.

In some ways, it's better to think of such a sum as running from $i = a$ to $i = b + 1$, but with the last item not quite included; that is, the sum covers every integer i that satisfies the inequality $a \le i < b + 1$, or in other words all of the integers in the interval $[a, b + 1)$. Of course, from this perspective, it's not the number b that matters but the number $b + 1$; if we call this B, then we can write $\sum_{a \le i < B}$ for what is normally written as $\sum_{i=a}^{b}$. Note also that it makes perfect sense to have $B = a$ (in other words, $b - a = -1$); then we are adding up no terms, and the sum is 0.

One nice consequence is that the number of terms in the sum is simply $B - a$ rather than $b - a + 1$. Perhaps more importantly, we have this theorem:

$$\sum_{A \le i < B} + \sum_{B \le i < C} = \sum_{A \le i < C},$$

which looks nicer than

$$\sum_{i=a}^{b} + \sum_{i=b+1}^{c} = \sum_{i=a}^{c}.$$

The upshot of all of this is that, when you see (for example) a sum as i runs from 2 to 5, you might want to think of it as a sum over $2 \le i < 6$ instead.

Some of the formulas for summing cubic polynomials are slightly simpler. With the traditional numbering, we have these (from pages 295 and 296 of the textbook):

$$\sum_{i=0}^{b} c = c(b + 1) \text{ if } c \text{ is constant};$$

$$\sum_{i=0}^{b} i = \frac{1}{2} b(b + 1) = \binom{b+1}{2};$$

$$\sum_{i=0}^{b} i^2 = \frac{1}{6} b(b + 1)(2b + 1);$$

$$\sum_{i=0}^{b} i^3 = \frac{1}{4} b^2 (b + 1)^2 = \binom{b+1}{2}^2.$$

(Here,

$$\binom{n}{r} = \frac{n!}{r!\,(n - r)!},$$

where $n! = n(n - 1)(n - 2) \cdots (3)(2)(1)$, is an expression that you don't need to learn if you don't want to but which is used in many mathematical formulas.)

With the off-by-1 numbering, we have these:

$$\sum_{0 \le i < B} c = cB \text{ if } c \text{ is constant};$$

$$\sum_{0 \le i < B} i = \frac{1}{2} B(B - 1) = \binom{B}{2};$$

$$\sum_{0 \le i < B} i^2 = \frac{1}{6} B(B - 1)(2B - 1);$$

$$\sum_{0 \le i < B} i^3 = \frac{1}{4} B^2 (B - 1)^2 = \binom{B}{2}^2.$$

Especially if you use $\binom{B}{2}$, then some of these are simpler.

It's also handy to have more general formulas starting at an arbitrary place rather than at $i = 0$ or 1. With the traditional numbering, we have these:

$$\sum_{i=a}^{b} c = c(b-a+1) \text{ if } c \text{ is constant;}$$
$$\sum_{i=a}^{b} i = \frac{1}{2}(a+b)(b-a+1);$$
$$\sum_{i=a}^{b} i^2 = \frac{1}{6}(2a^2+2ab+2b^2-a+b)(b-a+1);$$
$$\sum_{i=a}^{b} i^3 = \frac{1}{4}(a^2+b^2-a+b)(a+b)(b-a+1).$$

With the off-by-1 numbering, we have these:

$$\sum_{A\le i<B} c = c(B-A) \text{ if } c \text{ is constant;}$$
$$\sum_{A\le i<B} i = \frac{1}{2}(B-A)(A+B-1);$$
$$\sum_{A\le i<B} i^2 = \frac{1}{6}(B-A)(2A^2+2AB+2B^2-3A-3B+1);$$
$$\sum_{A\le i<B} i^3 = \frac{1}{4}(B-A)(A+B-1)(A^2+B^2-A-B).$$

These are now about equally complicated.

6.2 *Infinite series*

Besides this, we also consider *infinite* series, which are analogous to infinite improper integrals. Just as $\int_{x=a}^{\infty} f(x)\,\mathrm{d}x$ is defined as $\lim_{b\to\infty}\int_{x=a}^{b} f(x)\,\mathrm{d}x$, so an infinite series is defined as a limit of finite series:

$$\sum_{n=i}^{\infty} a_n = \lim_{j\to\infty}\sum_{n=i}^{j} a_n,$$

or equivalently $\lim_{J\to\infty}\sum_{i\le n<J} a_n$; the finite sum $\sum_{n=i}^{j} a_n$ (or $\sum_{i\le n<J} a_n$) is called a **partial sum** of the series. (As with infinite integrals, you can also replace i with $-\infty$, but we won't be doing that very often.) Now there is a limit (and hence Calculus) involved even for sequences. If this limit converges (to a finite real number), then we say that the infinite series **converges** (to that number); otherwise, it **diverges**. Sometimes it's useful to say that it diverges to ∞ or $-\infty$ (if it does), but this still counts as divergence.

You can also write

$$\sum_{n=i}^{\infty} a_n = \int_{x=i}^{\infty} a_{\lfloor x\rfloor}\,\mathrm{d}x;$$

that is, an infinite series isn't merely *analogous* to an infinite improper integral, it actually *is* an infinite improper integral, even if trying to evaluate this integral just turns it back into the series. Again, look at Figure 9.11.a on page 501 of the textbook; this time, ignore the function f and its blue curve, but notice how the area under the magenta staircase (which is the graph of $a_{\lfloor x\rfloor}$, so the area under it is the integral $\int_{x=1}^{\infty} a_{\lfloor x\rfloor}\,\mathrm{d}x$) represents the infinite sum $a_1 + a_2 + \cdots = \sum_{n=1}^{\infty} a_n$.

It's important to distinguish convergence of a series from convergence of its sequence of terms. If we think of the numbers a_0, a_1, a_2, and so on as forming a sequence $(a_0, a_1, a_2, \ldots)$, then this sequence converges if its limit $\lim_{n\to\infty} a_n$ exists; but if we think of them as the terms of a series, then this series converges if its sum $\sum_{n=0}^{\infty} a_n$ exists, and this is the limit of the sequence of partial sums, *not* the limit of the sequence of terms.

Nevertheless, there is a relationship between a series and its sequence of terms: the series can only converge if the sequence does, and in fact the series can only converge if the sequence of terms converges to zero! This is because the jth term is

$$a_j = \sum_{n=i}^{j} a_n - \sum_{n=i}^{j-1} a_n;$$

if the series converges, then

$$\lim_{j\to\infty} a_j = \sum_{n=i}^{\infty} a_n - \sum_{n=i}^{\infty} a_n = 0$$

(since $j - 1 \to \infty$ as $j \to \infty$), but if the series doesn't converge, then this argument is invalid and $\lim_{j\to\infty} a_j$ could be anything. Be careful, however, since this argument only goes one way; if the limit of the sequence of term *is* zero, then that tells you nothing about whether the series converges.

6.3 *The Fundamental Theorem for series*

In the analogy between sequences and functions, where differentiation of functions corresponds to differences of sequences and integrals correspond to series, there is an analogue of the Fundamental Theorem of Calculus. Just as $(\mathrm{d}/\mathrm{d}x)(\int_{t=a}^{x} f(t)\,\mathrm{d}t) = f(x)$ (the first part), so

$$\Delta_n \left(\sum_{m=i}^{n-1} a_m \right) = a_j.$$

And just as $\int_{x=a}^{b} (F'(x)) = F(b) - F(a)$ (the second part), so

$$\sum_{n=i}^{j-1} (\Delta_n b_n) = b_j - b_i.$$

(In each of these, I had to stop the sum short by 1; for the full analogy, you should really think of $\sum_{n=i}^{j-1}$ as $\sum_{i\le n<j}$, as described on page 38.)

The sum of a difference is called a **telescoping series**. A telescoping series converges precisely when the original sequence (not the difference) converges:

$$\sum_{n=i}^{\infty} (\Delta_n b_n) = \lim_{j\to\infty} \sum_{n=1}^{j-1} (\Delta_n b_n) = \lim_{j\to\infty} (b_j - b_i) = \lim_{j\to\infty} b_j - b_i.$$

This result is so important that I'll repeat it without the difference notation (which is not widely used):

$$\sum_{n=i}^{\infty} (b_{n+1} - b_n) = \lim_{j\to\infty} b_j - b_i.$$

Sometimes people prefer to write this as

$$\sum_{n=i}^{\infty} (b_n - b_{n-1}) = \lim_{j\to\infty} b_j - b_{i-1}.$$

Just as you can get a list of integrals that you can do by finding the derivatives of basic functions, so you can get a list of series that you can do by finding the differences of basic functions. We could do this with polynomials, for example; although it doesn't come out as simply as in the continuous case, you can derive formulas to sum any polynomial sequence. But an even simpler example is an exponential sequence. That is, consider the difference of r^n with respect to n, where r is constant.

$$\Delta_n(r^n) = r^{n+1} - r^n = r^n(r-1).$$

If anything, this is *simpler* than $\mathrm{d}(r^x)/\mathrm{d}x = r^x \ln x$; the natural logarithm has been replaced by a simple subtraction. Conversely, if you want to sum r^n, you just need to divide by the constant $r-1$. So

$$\sum_{i \le n < J} r^n = \frac{r^J - r^i}{r-1},$$

which is more commonly written as

$$\sum_{n=i}^{j} r^n = \frac{r^i - r^{j+1}}{1-r}.$$

Of course, this doesn't work if $r = 1$; for that, $\sum_{n=i}^{J} 1^n = J - i$, or $\sum_{n=i}^{j} 1^n = j - i + 1$.

A series like this is traditionally called a **geometric series**. The infinite version converges whenever $\lim_{J\to\infty} r^J$ exists (for $r \neq 1$), which happens precisely when $|r| < 1$, in which case the limit is actually 0. (If $r > 1$, then the limit is ∞; if $r = 1$, then the limit is $\lim_{J\to\infty} J = \infty$; if $r = -1$, then it oscillates between 1 and -1; and if $r < -1$, then it oscillates between ∞ and $-\infty$.) Therefore,

$$\sum_{n=i}^{\infty} r^n = -\frac{r^i}{r-1} = \frac{r^i}{1-r}$$

if $|r| < 1$.

6.4 *Convergence tests*

Here is a summary of all of the **convergence tests** that we use in this class. Every test has certain conditions under which it gives *no answer*, and then you'll have to try a different test. The first few terms are always irrelevant to convergence questions, so every condition only refers to what the terms do *eventually*: at some term a_j and then for every term a_k for $k \ge j$. (I'll write a for the sequence of terms of the series; that is, we are looking at

$$\sum_{n=i}^{\infty} a_n$$

for some integer i.)

Every convergence test, if it concludes that a series converges, gives a sequence of approximations of the sum of the series, along with an upper bound on the absolute value of the error of the approximations. Usually, however, we cannot compute the sum of the series exactly.

The definition

Even the definition of convergence can be viewed as a test. The sequence s in this test always exists; it's the sequence of partial sums in the definition. The problem, however, is that you might not be able to find a nice formula for it!

So, can you find a nice sequence s such that

$$s_m = \sum_{n=i}^{m} a_i$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does

$$\lim_{m\to\infty} s_m$$

exist (as a finite real number)? If not, then the series **diverges**. If so, then the series **converges**.

In fact,

$$\sum_{n=i}^{\infty} a_i = \lim_{m\to\infty} \sum_{n=i}^{m} a_i$$

when this limit converges (by definition).

The Telescoping Series Test

This is a slight variation of the definition that may be easier to spot. Can you find a nice sequence b such that

$$a_n = b_{n+1} - b_n$$

(eventually) or

$$a_n = b_n - b_{n+1}$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does the limit

$$\lim_{n\to\infty} b_n$$

converge (to a finite real number)? If not, then the series **diverges**. If so, then the series **converges**.

In fact,

$$\sum_{n=i}^{\infty} (b_{n+1} - b_n) = \lim_{n\to\infty} b_n - b_i$$

when this limit converges, and

$$\sum_{n=i}^{\infty} (b_n - b_{n+1}) = b_i - \lim_{n\to\infty} b_n$$

when this limit converges.

The Geometric Series Test

Can you write the series as

$$a_n = cr^n$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Is $c \neq 0$? If not, then the series **converges**. If so, then go on.

Is $|r| < 1$? If not, then the series **diverges**. If so, then the series **converges**.

In fact,

$$\sum_{n=i}^{\infty} cr^n = \frac{cr^i}{1-r}$$

when $|r| < 1$.

The nth-Term Test

This is probably the first test that you want to consider, unless the series fits one of the special forms above.

Does

$$\lim_{n\to\infty} a_n$$

converge to 0? If not, then the series **diverges**. If so, then this test gives **no answer**.

The Integral Test

Can you find a nice function f defined everywhere (eventually, say defined on $[j, \infty)$) such that $f(n) = a_n$ (eventually)? If not, then this test gives **no answer**. If so, then go on.

Does f take only nonnegative values (eventually)? If not, then this test gives **no answer**. If so, then go on.

Is f monotone decreasing (eventually)? If not, then this test gives **no answer**. If so, then go on.

Does

$$\int_j^\infty f(x)\,\mathrm{d}x$$

converge (to a finite real number, for some j)? If not, then the series **diverges**. If so, then the series **converges**.

In this case,

$$\sum_{n=i}^{m} f(n) + \int_m^\infty f(x)\,\mathrm{d}x \leq \sum_{n=i}^{\infty} f(n) \leq \sum_{n=i}^{m} f(n) + \int_{m+1}^\infty f(x)\,\mathrm{d}x.$$

The p-Series Test

Can you find a real number p such that

$$a_n = \frac{1}{n^p}$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Is $p > 1$? If not, then the series **diverges**. If so, then the series **converges**.

The Direct Comparison Test for Convergence

Does the series consist of only nonnegative terms (eventually)? If not, then this test gives **no answer**. If so, then go on.

Can you find a *convergent* series b such that

$$a_n \leq b_n$$

(eventually)? If not, then this test gives **no answer**. If so, then the original series a also **converges**.

The Direct Comparison Test for Divergence

Can you find a *divergent* series b such that

$$a_n \geq b_n$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does the series b consist of only nonnegative terms (eventually)? If not, then this test gives **no answer**. If so, then the original series a **diverges**.

The Limit Comparison Test

Does the series consist of only nonnegative terms (eventually)? If not, then this test gives **no answer**. If so, then go on.

Can you find a nice series b such that

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

converges to a *positive* real number? If not, then this test gives **no answer**. If so, then go on.

Does the series b converge? If not, then the original series a also **diverges**. If so, then the original series also **converges**.

The Absolute Convergence Test

Does the series

$$\sum_{n=i}^{\infty} |a_i|$$

of absolute values converge (to a finite real number)? If not, then this test gives **no answer**. If so, then the original series **converges**.

In this case, we say that the original series **converges absolutely**. If the original series converges (which we can only know by some other test) while the series of absolute values diverges, then the original series **converges conditionally**.

The Ratio Test

Does the limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

converge (to a finite real number)? If not, then this test gives **no answer**. If so, then go on.

Is this limit *different* from 1? If not, then this test gives **no answer**. If so, then go on.

Is this limit less than 1? If not, then the series **diverges**. If so, then the series **converges**.

The Root Test

Does the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

converge (to a finite real number)? If not, then this test gives **no answer**. If so, then go on.

Is this limit *different* from 1? If not, then this test gives **no answer**. If so, then go on.

Is this limit less than 1? If not, then the series **diverges**. If so, then the series **converges**.

The Alternating Series Test

Do we have either

$$a_n = (-1)^n |a_n|$$

or

$$a_n = -(-1)^n |a_n|$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Do we have

$$|a_{n+1}| \leq |a_n|$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does

$$\lim_{n\to\infty} |a_n|$$

converge to 0? If not, then the original series **diverges**. If so, then the original series **converges**.

In this case,

$$\sum_{n=i}^{m} a_n \leq \sum_{n=i}^{\infty} a_n \leq \sum_{n=i}^{m+1} a_n$$

if a_{m+1} is positive, and

$$\sum_{n=i}^{m+1} a_n \leq \sum_{n=i}^{\infty} a_n \leq \sum_{n=i}^{m} a_n$$

if a_{m+1} is negative.

Other tests

There are other tests (and some of these tests can be made more powerful too), but these tests (in these forms) are the only ones that you are responsible for knowing. In particular, every convergence problem in this class should succumb, one way or another, to at least one of these tests. However, there is no end to convergence tests, and mathematicians are still developing new ones, while some series have resisted all efforts so far!

7 Taylor series

One of the major applications of infinite series is to use series to approximate functions that are difficult to calculate. In this class, we mostly concentrate on series that approximate functions that you're already familiar with, because then I can assign you problems that have definite answers. (However, the really useful application is when you start with some other problem, such as an integral or a differential equation, that you *can't* work out exactly using the usual operations but which can be expressed as an infinite series.)

7.1 *Taylor polynomials*

Recall that when a function f is differentiable at a number a, then we can approximate f near a with a linear function that has both the same value and derivative as f does at a:

$$f(x) \approx L(x) = f(a) + f'(a)\,(x-a);$$

here, L is a linear function, $L(a) = f(a)$, and $L'(a) = f'(a)$. This is actually only the beginning (well, slightly after the beginning) of a whole sequence of approximations, each (typically) better than the one before it:

$$\begin{aligned}
f(x) &\approx P_0(x) = f(a);\\
f(x) &\approx P_1(x) = f(a) + f'(a)\,(x-a);\\
f(x) &\approx P_2(x) = f(a) + f'(a)\,(x-a) + \frac{1}{2}\,f''(a)\,(x-a)^2;\\
f(x) &\approx P_3(x) = f(a) + f'(a)\,(x-a) + \frac{1}{2}\,f''(a)\,(x-a)^2 + \frac{1}{6}\,f'''(a)\,(x-a)^3;\\
&\vdots
\end{aligned}$$

(The function that used to be called L is now called P_1.) The general form of this is

$$f(x) \approx P_k(x) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(a)(x-a)^n.$$

(Recall that $f^{(n)}$ is the nth derivative of f.) Of course, f must be differentiable at a at least k times for P_k to make sense.

The function P_k is the **Taylor polynomial** of f at a of order k. The Taylor polynomial of f at 0 of order k is also called the **Maclaurin polynomial** of f of order k. This terminology is standard (except for some variations in the phrase 'of order' that you may see); however, the notation P_k is *not* standard (and in principle it ought to mention f and a as well as k). Strictly speaking, Taylor polynomials are polynomial *functions* rather than polynomials as such (which are simply algebraic expressions without any variable picked out); otherwise, you'd have to mention the variable x as well.

Notice that a Taylor polynomial P_k of order k is a polynomial function of degree at most k. (The degree is normally exactly k, but it's smaller if $f^{(k)}(a)$ happens to be 0.) Also, the nth derivative of P_k at a agrees with that of f, if $n \le k$; that is,

$$P_k^{(n)}(a) = f^{(n)}(a)$$

if $n \le k$. (On the other hand, if $n > k$, then $P_k^{(n)}(a) = 0$, which is always the case for a higher-order derivative of a polynomial function when the order of the derivative is greater than the degree of the polynomial.) The Taylor polynomial of f at a of order k is the *only* polynomial function of degree at most k whose derivatives at a of order up to k agree with those of f.

Since polynomials are easy to work with, it's convenient to make approximations like these. But in practice, it's also important to know *how good* the approximations are. Since these approximations are based on the behaviour of f at a, we can really only expect them to be good when $x \approx a$. So one way to

say that these approximations work is to say that $P_k(x)$ approaches $f(x)$ (or more formally that the error of the approximation, $|P_k(x) - f(x)|$, approaches 0) as x approaches a. This is true for $k = 0$ if f is continuous at a, and for $k > 0$ if f is differentiable k times at a. But in fact, the higher-order Taylor polynomials satisfy a stronger condition:

$$\lim_{x \to a} \frac{|P_k(x) - f(x)|}{|x - a|^k} = 0,$$

which is called (one version of) **Taylor's Theorem**. As x approaches a, of course $|x - a|$ approaches zero, so dividing by $|x - a|$ would tend to make a positive quantity larger. So P_k is such a good approximation to f that the error not only approaches zero but still approaches zero even after dividing by $|x - a|$ several times.

When investigating these questions, it's helpful to change perspective slightly. Write R_k for $f - P_k$, the Taylor **remainder** of f at a of order k. Then the statement above, showing what a good approximation P_k is, becomes

$$\lim_{x \to a} \frac{|R_k(x)|}{|x - a|^k} = 0.$$

This is good to know, but it may not really be enough; it tells us that moving x close to a will make the approximation better, and very quickly; roughly, when x is already close to a, then moving it twice as close will make the approximation 2^k times better, or you can make the approximation one decimal digit more accurate by moving x only $\sqrt[k]{10}$ times as close. However, this doesn't tell us how accurate the approximation was to start with, nor how close x has to be for this method of improving the approximation to start working.

We can get better results if f is differentiable one more time ($k + 1$ times, not just k times) and near a (not just at a). This strong version of Taylor's Theorem says that

$$R_k(x) = \frac{(x - a)^{k+1}}{k!} \int_{t=0}^{1} (1 - t)^k f^{(k+1)}(a - at + xt)\, \mathrm{d}t,$$

as long as f is continuously differentiable $k + 1$ times (at least) between a and x. (The integral here may exist even if f is not *continuously* differentiable $k + 1$ times, but then the value of this integral might not equal the remainder.) To be more explicit, here is the statement for the first few values of k:

$$\begin{aligned}
f(x) &= f(a) + (x - a) \int_{t=0}^{1} f'(a - at + xt)\, \mathrm{d}t \\
&= f(a) + f'(a)(x - a) + (x - a)^2 \int_{t=0}^{1} (1 - t) f''(a - at + xt)\, \mathrm{d}t \\
&= f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \frac{(x - a)^3}{2} \int_{t=0}^{1} (1 - t)^2 f'''(a - at + xt)\, \mathrm{d}t \\
&\vdots
\end{aligned}$$

These statements may be proved by repeated application of integration by parts (and the Fundamental Theorem of Calculus, which is why $f^{(k+1)}$ must not only exist but also be continuous). To be specific, you can prove each statement using $u = (1 - t)^k / k!$ and $v = (x - a)^k f^{(k)}(a - at + xt)$, integrating by parts, simplifying, and (if applicable) applying the previous statement.

For purposes of approximation, it's useless to actually work out the integral that appears here; if you knew the exact value of $f^{(k+1)}$ at all of the points between a and x, then you could probably just evaluate f at x directly. However, if there is a value M_k such that you know that $f^{(k+1)}$ never has an absolute value greater than M_k at any point between a and x, then you can use M_k to get a bound on the remainder:

$$|R_k(x)| \leq \frac{M_k}{(k + 1)!} |x - a|^{k+1}.$$

The reason for this is that we know that $R_k(x)$ is exactly the integral that appeared in the full version of the theorem, and we can bound its absolute value using the bound on its integrand:

$$\begin{aligned}|R_k(x)| &= \left|\frac{(x-a)^{k+1}}{k!}\int_{t=0}^{1}(1-t)^k f^{(k+1)}(a-at+xt)\,\mathrm{d}t\right| \\ &\leq \frac{|x-a|^{k+1}}{k!}\int_{t=0}^{1}(1-t)^k\left|f^{(k+1)}(a-at+xt)\right|\mathrm{d}t \\ &\leq \frac{|x-a|^{k+1}}{k!}\int_{t=0}^{1}(1-t)^k M_k\,\mathrm{d}t = \frac{|x-a|^{k+1}}{k!}\frac{M_k}{k+1} = \frac{M_k}{(k+1)!}|x-a|^{k+1}.\end{aligned}$$

To be more specific:

$$|R_0(x)| = |f(x)-f(a)| \leq M_0|x-a|$$

if $|f'|$ is never greater than M_0 between a and x,

$$|R_1(x)| = \left|f(x)-\Big(f(a)+f'(a)(x-a)\Big)\right| \leq \frac{1}{2}M_1|x-a|^2$$

if $|f''|$ is never greater than M_1 between a and x,

$$|R_2(x)| = \left|f(x)-\left(f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2\right)\right| \leq \frac{1}{6}M_2|x-a|^3$$

if $|f'''|$ is never greater than M_2 between a and x, etc. Note that this upper bound on the absolute value of the remainder is basically the absolute value of the next term that you would add if you went one step further, except that instead of using a derivative at a, you must use the largest derivative (in absolute value) anywhere between a and x.

7.2 *Taylor series*

We can extend from polynomials to power series and get the **Taylor series** of f at a:

$$P_\infty(x) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(x-a)^n.$$

(When $a=0$, this is the **Maclaurin series** of f.) This power series exists as long as f is infinitely differentiable at a, that is as long as f has derivatives of all orders at a. However, there are no theorems guaranteeing that this series converges, nor that it's anything like $f(x)$ when it does converge (except that it must converge to $f(a)$ when $x=a$ exactly). We say that f is **analytic** at a if this series converges to $f(x)$ at least on some interval around a. Any function built out of the usual operations* is analytic, as long as it's infinitely differentiable, so everywhere that it is defined except where an absolute value or a root (or a power with a fractional exponent) is applied to 0 or an inverse trigonometric sine, cosine, secant, or cosecant is applied to ±1. However, there are functions for which the Taylor series exists but fails to converge (except when $x=a$ exactly); the only examples that I know are defined themselves as series, such as $f(x) = \sum_{n=0}^{\infty}\mathrm{e}^{-\sqrt{2^n}}\cos(2^n x)$ (which is not a power series but still converges everywhere by the Root Test). There are also functions for which the Taylor series converges but not to $f(x)$ (except when $x=a$ exactly); an example of this (with $a=0$) is $f(x) = \begin{cases}\mathrm{e}^{-x^2} & \text{for } x\neq 0,\\ 0 & \text{for } x=0.\end{cases}$

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* addition, subtraction, multiplication, division, taking opposites, taking reciprocals, taking absolute values, raising to the power of a constant, raising to a power when the base is positive, taking roots with a constant index, taking roots with a positive radicand, taking logarithms, the six trigonometric operations, and the six inverse trigonometric operations

There are several famous Taylor series of analytic functions that you should know:

$$x^k = \sum_{n=0}^{\infty} \binom{k}{n} (x-1)^n \text{ for } 0 < x < 2;$$

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!};$$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1} \text{ for } 0 < x \le 2;$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1};$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n};$$

$$\operatorname{atan} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } -1 \le x \le 1.$$

(You can check that these are Taylor series for the claimed functions by checking the functions' derivatives, and you can prove that these series converge for the claimed values of x using the usual convergence tests, but it takes more work to prove that they converge *to* the claimed functions. Much of this is proved in the textbook in Sections 9.7–9.10.)

The formula for x^k may seem particularly useless, and it mostly is when k is a whole number, but it is valid for any real number k, such as $k = -1$ (for $1/x$), $k = 1/2$ (for $\sqrt{x}$), etc. This formula includes $\binom{k}{n}$, the **binomial coefficient** of k with index n, which is defined by

$$\binom{k}{n} = \frac{k^{\underline{n}}}{n!} = \frac{k(k-1)(k-2)\cdots\big(k-(n-1)\big)}{n(n-1)(n-2)\cdots 1};$$

that is, the binomial coefficient is a fraction whose numerator and denominator each consists of n factors, with the denominator beginning at n to produce $n!$ and with the numerator beginning at k to produce $k^{\underline{n}}$, the **falling power** of k with index n (so in particular, $n! = n^{\underline{n}}$). Just as $0! = 1$, so $\binom{k}{0} = \frac{1}{1} = 1$; another useful fact is that $\binom{-1}{n} = (-1)^n$. (There is really a lot to be said about this stuff, which is part of the branch of mathematics called *combinatorics*, but the only thing that you're responsible for is to calculate $n!$ and $\binom{k}{n}$ for specific values of k and n.)

When you use these formulas, you may need to substitute some other expression for x, and you may need to start a sum at some other index. For example, if you want to evaluate

$$\sum_{n=3}^{\infty} \frac{x^n}{n},$$

then the important thing to notice is that the denominator is the same as the exponent (rather than the factorial of the exponent, as in some of the formulas) and that almost every natural number appears as an exponent (rather than only odd numbers or only even numbers, as in some of the formulas), which means that it's the formula for $\ln x$ that's relevant. To get the exponent in the right form, choose m so that $n = m + 1$; that is, $m = n - 1$. You now have

$$\sum_{m=2}^{\infty} \frac{x^{m+1}}{m+1}.$$

To get the right base, you might choose y so that $x = y - 1$; however, to get the factor of $(-1)^n$ as well, you should actually choose y so that $x = -(y-1)$. That is, $y = 1 - x$, so you now have

$$\sum_{m=2}^{\infty} \frac{\big(-(y-1)\big)^{m+1}}{m+1} = \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m+1}(y-1)^{m+1} = -\sum_{m=2}^{\infty} \frac{(-1)^{m}}{m+1}(y-1)^{m+1}.$$

Now you can match this against the formula for $\ln x$, using m in place of n and y in place of x, with an extra minus sign out front and with the first two terms missing. Since these missing terms are

$$\sum_{m=0}^{1} \frac{(-1)^{m}}{m+1}(y-1)^{m+1} = \frac{1}{1}(y-1)^{1} + \frac{-1}{2}(y-1)^{2} = -\frac{1}{2}y^2 + 2y - \frac{3}{2},$$

the original series equals $-\big(\ln y - (-1/2\, y^2 + 2y - 3/2)\big) = -\ln y - 1/2\, y^2 + 2y - 3/2$ whenever $0 < y \leq 2$. Remembering that $y = 1 - x$, you can finally conclude that

$$\sum_{n=3}^{\infty} \frac{x^n}{n} = -\ln(1-x) - \frac{1}{2}(1-x)^2 + 2(1-x) - \frac{3}{2} = -\ln(1-x) - \frac{1}{2}x^2 - x \text{ for } -1 \leq x < 1.$$

Some of these formulas appear in slightly different forms in the textbook; one version may be more convenient for a particular problem than another, but either version should suffice for all of the relevant problems.