9.8.27 If $f(x) = 1/x^2$, then it appears, by calculating the first few examples, that

$$f^{(n)}(x) = (-1)^n \frac{(n+1)!}{x^{n+2}}.$$

Certainly this is correct when $n = 0$, since

$$(-1)^0 \frac{(0+1)!}{x^{0+2}} = \frac{1}{x^2} = f(x).$$

So to verify the pattern, if this is correct when $n = k$, then

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(k)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^k \frac{(k+1)!}{x^{k+2}}\right) = (-1)^k \frac{(k+1)!\big(-(k+2)\big)}{x^{k+2+1}} = (-1)^{k+1} \frac{(k+2)!}{x^{k+3}},$$

so the pattern holds when $n = k+1$ as well. This shows (by 'mathematical induction') that the pattern is correct for every value of n.

Therefore,

$$f^{(n)}(1) = (-1)^n \frac{(n+1)!}{1^{n+2}} = (-1)^n \, (n+1)!,$$

so

$$\frac{f^{(n)}(1)}{n!} = \frac{(-1)^n \, (n+1)!}{n!} = (-1)^n (n+1),$$

and the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$