Convergence tests

This is a summary of the **convergence tests** that we use in this class. Every test has certain conditions under which it gives *no answer*, and then you'll have to try a different test. The first few terms are always irrelevant to convergence questions, so every condition only refers to what the terms do *eventually*: at some term a_j and then for every term a_k for $k \geq j$. (I'll write a for the sequence of terms of the series; that is, we are looking at

$$\sum_{n=i}^{\infty} a_n$$

for some integer i.)

Every convergence test, if it concludes that a series converges, gives a sequence of approximations of the sum of the series, along with an upper bound on the absolute value of the error of the approximations. Usually, however, we cannot compute the sum of the series exactly.

The definition

Even the definition of convergence can be viewed as a test. The sequence s in this test always exists; it's the sequence of partial sums in the definition. The problem, however, is that you might not be able to find a nice formula for it!

So, can you find a nice sequence s such that

$$s_m = \sum_{n=i}^{m} a_i$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does

$$\lim_{m \to \infty} s_m$$

exist (as a finite real number)? If not, then the series **diverges**. If so, then the series **converges**.

In fact,

$$\sum_{n=i}^{\infty} a_i = \lim_{m \to \infty} \sum_{n=i}^{m} a_i$$

when this limit converges (by definition).

The Telescoping Series Test

This is a slight variation of the definition that may be easier to spot. Can you find a nice sequence b such that

$$a_n = b_{n+1} - b_n$$

(eventually) or

$$a_n = b_n - b_{n+1}$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does the limit

$$\lim_{n \to \infty} b_n$$

converge (to a finite real number)? If not, then the series **diverges**. If so, then the series **converges**.

In fact,

$$\sum_{n=i}^{\infty} (b_{n+1} - b_n) = \lim_{n \to \infty} b_n - b_i$$

when this limit converges, and

$$\sum_{n=i}^{\infty} (b_n - b_{n+1}) = b_i - \lim_{n \to \infty} b_n$$

when this limit converges.

The Geometric Series Test

Can you write the series as

$$a_n = cr^n$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Is $c \neq 0$? If not, then the series **converges**. If so, then go on.

Is $|r| < 1$? If not, then the series **diverges**. If so, then the series **converges**.

In fact,

$$\sum_{n=i}^{\infty} cr^n = \frac{cr^i}{1-r}$$

when $|r| < 1$.

The nth-Term Test

This is probably the first test that you want to consider, unless the series fits one of the special forms above.

Does

$$\lim_{n \to \infty} a_n$$

converge to 0? If not, then the series **diverges**. If so, then this test gives **no answer**.

The Integral Test

Can you find a nice function f defined everywhere (eventually, say defined on $[j, \infty)$) such that $f(n) = a_n$ (eventually)? If not, then this test gives **no answer**. If so, then go on.

Does f take only nonnegative values (eventually)? If not, then this test gives **no answer**. If so, then go on.

Is f monotone decreasing (eventually)? If not, then this test gives **no answer**. If so, then go on.

Does

$$\int_j^{\infty} f(x)\,\mathrm{d}x$$

converge (to a finite real number, for some j)? If not, then the series **diverges**. If so, then the series **converges**.

In this case,

$$\sum_{n=i}^{m} f(n) + \int_m^{\infty} f(x)\,\mathrm{d}x \leq \sum_{n=i}^{\infty} f(n) \leq \sum_{n=i}^{m} f(n) + \int_{m+1}^{\infty} f(x)\,\mathrm{d}x.$$

The p-Series Test

Can you find a real number p such that

$$a_n = \frac{1}{n^p}$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Is $p > 1$? If not, then the series **diverges**. If so, then the series **converges**.

The Direct Comparison Test for Convergence

Does the series consist of only nonnegative terms (eventually)? If not, then this test gives **no answer**. If so, then go on.

Can you find a *convergent* series b such that

$$a_n \leq b_n$$

(eventually)? If not, then this test gives **no answer**. If so, then the original series a also **converges**.

The Direct Comparison Test for Divergence

Can you find a *divergent* series b such that

$$a_n \geq b_n$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does the series b consist of only nonnegative terms (eventually)? If not, then this test gives **no answer**. If so, then the original series a **diverges**.

The Limit Comparison Test

Does the series consist of only nonnegative terms (eventually)? If not, then this test gives **no answer**. If so, then go on.

Can you find a nice series b such that

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

converges to a *positive* real number? If not, then this test gives **no answer**. If so, then go on.

Does the series b converge? If not, then the original series a also **diverges**. If so, then the original series also **converges**.

The Alternating Series Test

Do we have either

$$a_n = (-1)^n |a_n|$$

or

$$a_n = -(-1)^n |a_n|$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Do we have

$$|a_{n+1}| \leq |a_n|$$

(eventually)? If not, then this test gives **no answer**. If so, then go on.

Does

$$\lim_{n\to\infty} |a_n|$$

converge to 0? If not, then the original series **diverges**. If so, then the original series **converges**.

In this case,

$$\sum_{n=i}^{m} a_n \leq \sum_{n=i}^{\infty} a_n \leq \sum_{n=i}^{m+1} a_n$$

if a_{m+1} is positive, and

$$\sum_{n=i}^{m+1} a_n \leq \sum_{n=i}^{\infty} a_n \leq \sum_{n=i}^{m} a_n$$

if a_{m+1} is negative.

The Ratio Test

Does the limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

converge (to a finite real number)? If not, then this test gives **no answer**. If so, then go on.

Is this limit *different* from 1? If not, then this test gives **no answer**. If so, then go on.

Is this limit less than 1? If not, then the series **diverges**. If so, then the series **converges**.

The Root Test

Does the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

converge (to a finite real number)? If not, then this test gives **no answer**. If so, then go on.

Is this limit *different* from 1? If not, then this test gives **no answer**. If so, then go on.

Is this limit less than 1? If not, then the series **diverges**. If so, then the series **converges**.

The Absolute Convergence Test

Does the series

$$\sum_{n=i}^{\infty} |a_i|$$

of absolute values converge (to a finite real number)? If not, then this test gives **no answer**. If so, then the original series **converges**.

In this case, we say that the original series **converges absolutely**. If the original series converges (which we can only know by some other test) while the series of absolute values diverges, then the original series **converges conditionally**.

Other tests

There are other tests (and some of these tests can be made more powerful too), but these tests (in these forms) are the only ones that you are responsible for knowing. In particular, every convergence problem in this class should succumb, one way or another, to at least one of these tests. However, there is no end to convergence tests, and mathematicians are still developing new ones, while some series have resisted all efforts so far!