Sequences and series

A **sequence** is a function whose domain consists only of integers. (It's not necessary that all integers belong to the domain, just that nothing else does.) To emphasize that we're considering a sequence, people often write f_n instead of $f(n)$ when f is a sequence (and n is an integer in its domain). In fact, 'f' is not a very common name for a sequence; 'a' and 'x' (or letters near them) are much more common. Similarly, the argument of a sequence is usually denoted by a letter near the middle of the alphabet (usually between 'i' and 'n'), since these letters are often used for integers. (Still, as with any other variable, you can use any letter that you like in principle.) There is also some redundant terminology: instead of speaking of the input (or argument) and output (or value) of a function, we speak of an **index** and **term** of a sequence. For example, if $a_n = (-2)^n$, then the term with index 3 is $a_3 = (-2)^3 = -8$. (Sometimes people say that 8 is the 3rd term, but this really only works if a_n is undefined when $n < 1$.)

Since Calculus is about continuously varying quantities and a sequence has only discrete values (at most one for each integer), there's not much Calculus to be done with a sequence. Nevertheless, there is some: you can consider the limit of a sqeuence approaching infinity (or negative infinity). That is, while $\lim_{n \to c} a_n$ (for finite c), $\frac{\mathrm{d}a_n}{\mathrm{d}n}$, and $\int a_n \,\mathrm{d}n$ don't make sense, nevertheless $\lim_{n \to \infty} a_n$ and $\lim_{n \to -\infty} a_n$ can make sense. I'll focus on the first of these, which you can call simply the **limit** of the sequence, because many of our sequences will only be defined at natural numbers; however, limits approaching negative infinity really aren't much different.

Sometimes it's convenient to think of a sequence as the restriction to integers of some more general function. For example, if you're working with the sequence $a_n = 3n^2$, then you can think of the function $f(x) = 3x^2$; while f is defined for all real numbers and a is defined only for integers, otherwise they are the same thing. Since $\lim_{x \to \infty} f(x) = \infty$, this tells us that $\lim_{n \to \infty} a_n = \infty$ too. So most of the time, you can work out the limit of a sequence in the same way that you work out any other limit approaching infinity. If $a_n = f(n)$ for n an integer and f has a limit (possibly infinite) approaching infinity, then a has the same limit; this is a theorem. However, it's possible that a has a limit even when f does not, for example if $f(x) = \sin(\pi x)$. This has no limit as $x \to \infty$, since all values between -1 and 1 are taken for arbitrarily large values of x. When n is an integer, however, $\sin(\pi n) = 0$, so the limit of the sequence $a_n = \sin(\pi n)$ (which is really just the sequence $a_n = 0$) is 0.

There are some more systematic ways of turning a sequence into a function that's defined everywhere (or almost everywhere). These involve the floor and ceiling operations: the **floor** $\lfloor x \rfloor$ of a real number x is the largest integer that's not larger than x, and the **ceiling** $\lceil x \rceil$ of x is the smallest integer that's not smaller than x. Ever since you first learnt to round numbers up and down, you've been using these operations, even if you didn't have names for them; for example, $\lfloor 2.37 \rfloor = 2$ (round down to the nearest integer), and $\lceil 2.37 \rceil = 3$ (round up to the nearest integer). An important inequality about floors and ceilings is

$$\lfloor x \rfloor \le x \le \lceil x \rceil.$$

As long as x is itself fractional (that is not an integer), then

$$\lfloor x \rfloor < x < \lceil x \rceil,$$

and in that case you also have

$$\lfloor x \rfloor + 1 = \lceil x \rceil.$$

(But integers are an exception; if x is an integer, then $\lfloor x \rfloor$, $\lceil x \rceil$, and x are all equal to each other.)

Using these, we can convert any sequence into a function defined more generally: if a is a sequence, then we can consider $a_{\lfloor x \rfloor}$ and $a_{\lceil x \rceil}$. If a is defined for all integers, then these will be defined for all values of x; even if a isn't defined for all integers, still $a_{\lfloor x \rfloor}$ and $a_{\lceil x \rceil}$ will be defined for many more real numbers. And now we have this theorem:

$$\lim_{x \to \infty} a_{\lfloor x \rfloor} = \lim_{n \to \infty} a_n = \lim_{x \to \infty} a_{\lceil x \rceil}.$$

These functions $a_{\lfloor x \rfloor}$ and $a_{\lceil x \rceil}$ are unusual, since they are (for most sequences) discontinuous at every integer, but they can be handy to think about.

You can see a picture of these in Figure 9.11 on page 508 of the textbook. (The textbook is using this picture for a different purpose, although it is related, as you'll see later on.) In this picture, they begin with a function f and then construct a sequence a out of it by defining $a_n = f(n)$. Then on the top (Figure 9.11.a), they draw the graph of $y = f(x)$ in blue along with a graph of $y = a_{\lfloor x \rfloor} = f(\lfloor x \rfloor)$ in magenta; on the bottom (Figure 9.11.b), they draw a graph of $y = f(x)$ in blue again but now with a graph of $y = a_{\lceil x \rceil} = f(\lceil x \rceil)$ in magenta. You'll notice that the sequence and all three of the other functions tend to the same limit (which in this case is 0).

Series

I wrote above that you can't do much Calculus on sequences; in particular, I remarked that the derivative $\frac{\mathrm{d}a_n}{\mathrm{d}n}$ and integral $\int a_n \,\mathrm{d}n$ don't make sense. Ultimately, this is because $\mathrm{d}n$, an infinitesimal (infinitely small) but non-zero change in n, doesn't make sense when n takes only integer values; the smallest possible non-zero change in n is a change by 1, which is not infinitely small.

But there is something *analogous* to derivatives and integrals. The analogue to derivatives is the **difference** $\Delta a_n = a_{n+1} - a_n$. Whereas the derivative is defined as a limit of difference quotients, the difference simply *is* a difference quotient where the change in n is $\Delta n = 1$. (Strictly speaking, this symbol 'Δ' should only come before the name of a sequence to define a new sequence; Δa_n means $(\Delta a)_n$, not $\Delta(a_n)$. In practice, people write it in front of any expression involving n, as I did just a bit ago in writing Δn, but this only works if you fix an independent variable n ahead of time. Unfortunately, sequences do not have an analogue of the differential that will automatically take care of changing from one variable to another.)

The analogue to an integral is a **series**, which is the result of adding up some of the terms of a sequence. (This word can be confusing, in two ways. The first is a quirk of grammar: the plural of 'series' is just 'series' again. You can say 'serieses' as the plural, although this is nonstandard, but using 'serie' as the singular is just plain wrong. The other confusing thing is that, in ordinary language, 'sequence' and 'series' mean basically the same thing; but in mathematics, a sequence is the more basic concept, and a series is a sum that you build out of a sequence.)

Like differences, a finite series has no Calculus in it; you just add up some numbers. For example,

$$\sum_{n=3}^{7} n^2 = (3)^2 + (4)^2 + (5)^2 + (6)^2 + (7)^2 = 135.$$

Strictly speaking, this is analogous to a proper integral such as $\int_{x=3}^{8} x^2 \,\mathrm{d}x$. Actually, this is more than just an analogy: a series *is* an integral, albeit one whose Calculus content is trivial. Specifically,

$$\sum_{n=i}^{j} a_n = \int_{x=i}^{j+1} a_{\lfloor x \rfloor} \,\mathrm{d}x = \int_{x=i-1}^{j} a_{\lceil x \rceil} \,\mathrm{d}x.$$

Since these are integrals of piecewise-constant functions, working them out is easy and just results in the original sum. So you don't want to evaluate a series by turning it into an integral; still, it can be handy to know that this can be done, becuase we know a lot of theorems about integrals that now automatically apply to series.

Besides this, we also consider *infinite* series, which are analogous to infinite improper integrals. Just as $\int_{x=a}^{\infty} f(x) \,\mathrm{d}x$ is defined as $\lim_{b \to \infty} \int_{x=a}^{b} f(x) \,\mathrm{d}x$, so an infinite series is defined as a limit of finite series:

$$\sum_{n=i}^{\infty} a_n = \lim_{j \to \infty} \sum_{n=i}^{j} a_n.$$

(As with infinite integrals, you can also replace i with $-\infty$, but we won't be doing that very often.) Now there is a limit (and hence Calculus) involved even for sequences. If this limit converges (to a finite real number), then we say that the infinite series **converges** (to that number); otherwise, it **diverges**. Sometimes it's useful to say that it diverges to ∞ or $-\infty$ (if it does), but this still counts as divergence.

You can also write

$$\sum_{n=i}^{\infty} a_n = \int_{x=i}^{\infty} a_{\lfloor x \rfloor}\,\mathrm{d}x;$$

that is, an infinite series isn't merely analogous to an infinite improper integral, it actually *is* an infinite improper integral, even if trying to evaluate this integral just turns it back into the series. Again, look at Figure 9.11.a on page 508 of the textbook; this time, ignore the function f and its blue curve, but notice how the area under the magenta staircase (which is the graph of $a_{\lfloor x \rfloor}$, so the area under it is the integral $\int_{x=1}^{\infty} a_{\lfloor x \rfloor}\,\mathrm{d}x$) represents the infinite sum $a_1 + a_2 + \cdots = \sum_{n=1}^{\infty} a_n$.