Multiplying vectors

The last handout discussed adding and subtracting vectors and multiplying them by scalars. This one discusses various ways of multiplying them by each other.

Projections

If you have two vectors $\mathbf{u}$ and $\mathbf{v}$, and assuming that neither of them is $\mathbf{0}$, place them so that they both start at the same point A and then draw a line from $A+\mathbf{v}$ to the line through A and $A+\mathbf{u}$ so that these lines intersect at a right angle. Let B be the point where these lines intersect; the vector $B-A$ is the **projection** of $\mathbf{v}$ onto $\mathbf{u}$, denoted $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$. Sometimes people also consider the projection of $\mathbf{v}$ *perpendicular* to $\mathbf{u}$; this is the vector from $A+\mathbf{u}$ to B:

$$\operatorname{proj}_{\mathbf{u}}^{\perp} \mathbf{v} = \mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}.$$

(In general, the symbol '$\perp$' is used when talking about perpendicular things, which the shape of the symbol is supposed to remind you of.)

A related concept is the **component** of $\mathbf{v}$ in the direction of $\mathbf{u}$, denoted $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$; this is a scalar chosen so that

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \operatorname{comp}_{\mathbf{u}} \mathbf{v}\, \hat{\mathbf{u}}.$$

It's a common mistake to think that $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ has the same direction as $\mathbf{u}$, so that consequently $\operatorname{comp}_{\mathbf{u}} \mathbf{v} = \|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|$. But in fact, $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ can just as easily have the opposite direction, so the general rule is

$$\operatorname{comp}_{\mathbf{u}} \mathbf{v} = \pm \|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|.$$

The component of $\mathbf{v}$ in the direction of $\mathbf{u}$ is positive if $\mathbf{u}$ and $\mathbf{v}$ have roughly the same direction but negative if they have roughly opposite directions. (It's also possible that this component is zero, when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.)

I have not allowed $\mathbf{v}$ to be the zero vector, because then $A+\mathbf{v}$ is simply A, right on the line through A and $A+\mathbf{u}$, so it makes no sense to draw anything from that point perpendicular to that line. However, since we're already on the line, we can simply take B to be A as well, so that $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$, which is $B-A$, is also $\mathbf{0}$. Thus, we have these results:

$$\operatorname{proj}_{\mathbf{u}} \mathbf{0} = \mathbf{0},\ \operatorname{comp}_{\mathbf{u}} \mathbf{0} = 0.$$

Now $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ and $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$ exist no matter what $\mathbf{v}$ is (although it's still necessary that $\mathbf{u} \neq \mathbf{0}$). Once we have that, you can verify these facts by drawing the relevant pictures:

$$\operatorname{proj}_{\mathbf{u}} (\mathbf{v}+\mathbf{w}) = \operatorname{proj}_{\mathbf{u}} \mathbf{v} + \operatorname{proj}_{\mathbf{u}} \mathbf{w}, \text{ so } \operatorname{comp}_{\mathbf{u}} (\mathbf{v}+\mathbf{w}) = \operatorname{comp}_{\mathbf{u}} \mathbf{v} + \operatorname{comp}_{\mathbf{u}} \mathbf{w};$$
$$\operatorname{proj}_{\mathbf{u}} (a\mathbf{v}) = a \operatorname{proj}_{\mathbf{u}} \mathbf{v}, \text{ so } \operatorname{comp}_{\mathbf{u}} (a\mathbf{v}) = a \operatorname{comp}_{\mathbf{u}} \mathbf{v}.$$

This is all well and good, but if you know a little trigonometry, then you can get a nice formula for this component. This is because $\mathbf{v}$ forms the hypotenuse of a right triangle, one of whose legs is $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$, and whose angle next to that leg is $\angle(\mathbf{u},\mathbf{v})$ if $\mathbf{u}$ and $\mathbf{v}$ have roughly the same direction or $\pi - \angle(\mathbf{u},\mathbf{v})$ if they have roughly opposite directions. In the first case,

$$\cos\angle(\mathbf{u},\mathbf{v}) = \frac{\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|};$$

in the other case,

$$\cos\angle(\mathbf{u},\mathbf{v}) = -\cos\Big(\pi - \angle(\mathbf{u},\mathbf{v})\Big) = -\frac{\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|}{\|\mathbf{v}\|} = -\frac{-\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|} = \frac{\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|}.$$

In the middle, when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $\cos\angle(\mathbf{u},\mathbf{v})$ and $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$ are both 0. So in any case,

$$\operatorname{comp}_{\mathbf{u}} \mathbf{v} = \|\mathbf{v}\| \cos\angle(\mathbf{u},\mathbf{v})$$

as long as $\mathbf{v} \neq \mathbf{0}$. (If $\mathbf{v} = \mathbf{0}$, then the angle $\angle(\mathbf{u},\mathbf{v})$ doesn't make sense, but the equation is still true in a way, since it becomes the true statement $0 = 0$ no matter what value you use for the angle.) We saw earlier how to express this cosine using only $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u}-\mathbf{v}\|$, but for now, let's just leave it as $\cos\angle(\mathbf{u},\mathbf{v})$.

The dot product

This now suggests that we'll get a very nice operation if we define

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \operatorname{comp}_{\mathbf{u}} \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \angle(\mathbf{u}, \mathbf{v}).$$

This has many nice properties; for example, these follow from the corresponding properties for components:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w}),$$
$$\mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v}).$$

However, since $\mathbf{u}$ and $\mathbf{v}$ appear symmetrically in the formula with the cosine, we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u},$$

and then these properties also follow:

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w}),$$
$$(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}).$$

The definition $\|\mathbf{u}\| \operatorname{comp}_{\mathbf{u}} \mathbf{v}$ allows $\mathbf{v}$ to be $\mathbf{0}$, but not $\mathbf{u}$. However, since the operation is symmetric when the vectors are nonzero, we can define it so that it continues to be symmetric, so that $\mathbf{0} \cdot \mathbf{v} = 0$ as well as $\mathbf{v} \cdot \mathbf{0} = 0$. In particular, we define $\mathbf{0} \cdot \mathbf{0}$ to be 0. (Thus, it remains true in a way that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \angle(\mathbf{u}, \mathbf{v})$, even when $\angle(\mathbf{u}, \mathbf{v})$ doesn't make sense, because in that case the equation becomes $0 = 0$ no matter what value you use for the angle.) Then the properties listed above continue to be true.

By this point, you should see where the notation comes from; this operation looks a lot like multiplication. It's variously called **inner multiplication** (for the operation) or the **inner product** (for the result of the operation), the **scalar product** (because the result is a scalar), or (naming it after its notation) the **dot product**. (Don't confuse *scalar multiplication*, describing the operation for $a\mathbf{v}$, with the *scalar product*, describing the result of the operation $\mathbf{u} \cdot \mathbf{v}$.) The properties above state that the dot product distrubutes over addition, that it's commutative, associative with scalar multiplication, etc.

Since angles can be expressed in terms of lengths, so can the dot product; you get

$$\mathbf{u} \cdot \mathbf{v} = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{2},$$

an expression that works regardless of whether $\mathbf{u}$ and $\mathbf{v}$ are nonzero. An important special case is when $\mathbf{u}$ and $\mathbf{v}$ are the same vector; then this simplifies to

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2.$$

(Another way to see this is that the angle between a vector and itself is 0, the cosine of which is 1, so $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\| \, \|\mathbf{v}\| \cos 0 = \|\mathbf{v}\|^2$.)

However, as a practical matter, there is a better way to calculate this. Because the dot product distributes over addition and associates with scalar multiplication, we only need to know $\mathbf{i} \cdot \mathbf{i}$, $\mathbf{i} \cdot \mathbf{j}$, and so on; that is, we only need to know what it does to the standard basis vectors. Since these vectors are all perpendicular to one another, so the cosine between any two different ones is 0, these dot products are almost all 0. The exception is the dot product of one of these with itself; since these vectors all have a magnitude of 1, the dot product of any one with itself is $1^2 = 1$. So in 2 dimensions,

$$\langle a, b \rangle \cdot \langle c, d \rangle = (a\mathbf{i} + b\mathbf{j}) \cdot (c\mathbf{i} + d\mathbf{j}) = ac\,\mathbf{i} \cdot \mathbf{i} + ad\,\mathbf{i} \cdot \mathbf{j} + bc\,\mathbf{j} \cdot \mathbf{i} + bd\,\mathbf{j} \cdot \mathbf{j} = ac\,1 + ad\,0 + bc\,0 + bd\,1 = ac + bd;$$

in 3 dimensions,

$$\langle a, b, c \rangle \cdot \langle d, e, f \rangle = ad + be + cf$$

by a similar calculation, and most generally in n dimensions,

$$\langle a_1, a_2, \ldots, a_n \rangle \cdot \langle b_1, b_2, \ldots, b_n \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

That is, you multiply corresponding compoenents of the vectors and add these all up. For example,

$$\langle 1, -2 \rangle \cdot \langle 3, 5 \rangle = (1)(3) + (-2)(5) = 3 - 10 = -7.$$

Now its best to give formulas for angles, projections, and components in terms of the dot product, rather than the other way around. So:

$$\begin{aligned}
\operatorname{comp}_{\mathbf{u}} \mathbf{v} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}; \\
\operatorname{proj}_{\mathbf{u}} \mathbf{v} &= \operatorname{comp}_{\mathbf{u}} \mathbf{v}\, \hat{\mathbf{u}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}; \\
\angle(\mathbf{u}, \mathbf{v}) &= \operatorname{acos} \frac{\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|} = \operatorname{acos} \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.
\end{aligned}$$

Even lengths can be expressed using the dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

Row vectors

I developed the dot product geometrically, and we've seen that it's closely related to lengths and angles. I remarked before that lengths and angles don't always make sense, and the same goes for the the dot product (as well as projections and components onto a given vector). For example, if x is measured in seconds (s) and y is measured in metres (m), then $\langle 1\,\mathrm{s}, -2\,\mathrm{m} \rangle \cdot \langle 3\,\mathrm{s}, 5\,\mathrm{m} \rangle = 3\,\mathrm{s}^2 - 10\,\mathrm{m}^2$ doesn't really make sense.

On the other hand, sometimes dot products can make sense in a context like this. For example, suppose that x represents the time at which something occurs and y represents its location, so that the vector $\Delta P = \langle \Delta x, \Delta y \rangle$ represents a passage of time together with a change in location, like the vectors above might do; then if the object in question is a missile that's going to explode at some unkown time and distance and you think that it's going to move slowly while I think that it's going to move quickly, then we might make a bet where I pay you \$1 for every second that it lasts until it explodes but you pay me \$2 for every metre that it travels. If it travels 5 metres in 3 seconds before exploding, then you'll get $(1)(3) - (2)(5) = -7$ dollars, or put another way, you'll owe me \$7. This can be represented as the dot product

$$\langle \$1/\mathrm{s}, -\$2/\mathrm{m} \rangle \cdot \langle 3\,\mathrm{s}, 5\,\mathrm{m} \rangle = (\$1/\mathrm{s})(3\,\mathrm{s}) + (-\$2/\mathrm{m})(5\,\mathrm{m}) = \$3 - \$10 = -\$7,$$

where the first vector is determined by the nature of our bet (you get \$1 per second and pay \$2 per metre), while the second vector is determined by the behaviour of the missile (it lasts 3 seconds and travels 5 metres).

Now, while the vector $\langle 3\,\mathrm{s}, 5\,\mathrm{m} \rangle$ really does describe a change in x and a change in y, where x and y represent time and position as I stated above, the vector $\langle \$1/\mathrm{s}, -\$2/\mathrm{m} \rangle$ does not. In the context of measuring time and position, this vector is a different kind of vector, one for which a dot product with an ordinary vector makes sense, even though lengths and angles don't make sense for any of these vectors. A vector like this is variously called a **dual vector**, a **covector**, or a **row vector**; in the last case, an ordinary vector may be called a **column vector**. I'll use the terminology of row and column vectors, which ultimately comes from matrix theory.

Row vectors obey the same rules of arithmetic as column vectors; here is a list of operations with these that make sense:

- Addition: adding a column vector to a point to get another point, adding two column vectors together to get another column vector, adding two row vectors together to get another row vector;
- Subtraction: subtracting a column vector from a point to get another point, subtracting one column vector from another to get another column vector, subtracting one row vector from another to get another row vector;
- Multiplication: multiplying a column vector by a scalar to get another column vector, multiplying a row vector by a scalar to get another row vector, multiplying a row vector and a column vector to get a scalar.

In particular, there is (in general) no notion of 'row point' that can interact with row vectors in the way that points interact with column vectors.

Area

Now let's go back to a geometric conception of vectors. If you take two vectors $\mathbf{u}$ and $\mathbf{v}$ and place them to start at a point A, then you can connect their endpoints to make a triangle and then ask what the area of that triangle is. It's actually a bit nicer to think of that triangle as half of a parallelogram: two opposite sides of the parallelogram are $\mathbf{u}$, one running from A to $A+\mathbf{u}$, the other running from $A+\mathbf{v}$ to $A+\mathbf{v}+\mathbf{u}$; the other two opposite sides are $\mathbf{v}$, one running from A to $A+\mathbf{v}$, the other running from $A+\mathbf{u}$ to $A+\mathbf{u}+\mathbf{v}$ (which of course is the same as $A+\mathbf{v}+\mathbf{u}$).

This question can be asked in any number of dimensions, and the answer may be written $\|\mathbf{u}\times\mathbf{v}\|$. This notation suggests that this area will be the magnitude of something more fundamental, which is $\mathbf{u}\times\mathbf{v}$ itself, and this is true to an extent, but exactly how that works depends on how many dimensions we're in. So for now, I'm just going to stick with $\|\mathbf{u}\times\mathbf{v}\|$. However, I can give you the terminology: whatever $\mathbf{u}\times\mathbf{v}$ is, the operation may be called **outer multiplication**, and the result may be called the **outer product** or the **cross product**, and in 3 dimensions (where it is best known), it's also called the **vector product**.

With the help of trigonometry,

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\angle(\mathbf{u},\mathbf{v}).$$

Notice that this sine is always positive, since the angle lies between 0 and π. For such an angle θ, $\sin\theta = \sqrt{1-\cos^2\theta}$; with the help of the dot product, this means that

$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2}.$$

(This formula makes sense even if $\mathbf{u}$ or $\mathbf{v}$ is the zero vector, in which case the result is zero.) If you write out $\mathbf{u}\cdot\mathbf{v}$ in this expression in terms of the lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u}-\mathbf{v}\|$, then the formula factors as

$$\|\mathbf{u}\times\mathbf{v}\| = \frac{\sqrt{-(\|\mathbf{u}\|+\|\mathbf{v}\|+\|\mathbf{u}-\mathbf{v}\|)(\|\mathbf{u}\|+\|\mathbf{v}\|-\|\mathbf{u}-\mathbf{v}\|)(\|\mathbf{u}\|-\|\mathbf{v}\|+\|\mathbf{u}-\mathbf{v}\|)(\|\mathbf{u}\|-\|\mathbf{v}\|-\|\mathbf{u}-\mathbf{v}\|)}}{2}.$$

(Despite the initial minus sign, the expression inside the square root is positive, since the last factor is negative.) This result was known to the ancient Greek–Egyptian mathematician and inventor Hero (or Heron) of Alexandria. (He invented the steam engine, the windmill, and the vending machine, although none of these caught on at the time.)

If $\mathbf{u}$ and $\mathbf{v}$ are parallel (or antiparallel), or if either (or both) of them is the zero vector $\mathbf{0}$, then $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\,\|\mathbf{v}\|$, so $\|\mathbf{u}\times\mathbf{v}\| = 0$. From another perspective, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, then the angle between them is 0, whose sine is 0; if they're antiparallel, then the sine is still $\sin\pi = 0$. In this case, you don't really have a parallelogram, but a simple line segment (or a point if $\mathbf{u}$ and $\mathbf{v}$ are both $\mathbf{0}$), whose area is indeed zero.

Here are some important algebraic properties of $\|\mathbf{u}\times\mathbf{v}\|$:

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{v}\times\mathbf{u}\|;$$

$$\|\mathbf{u}\times a\mathbf{v}\| = |a|\,\|\mathbf{u}\times\mathbf{v}\|;$$

$$\|\mathbf{u}\times\mathbf{v}\| = \left\|\mathbf{u}\times\operatorname{proj}_{\mathbf{u}}^{\perp}\mathbf{v}\right\| = \|\mathbf{u}\|\,\left\|\operatorname{proj}_{\mathbf{u}}^{\perp}\mathbf{v}\right\|.$$

(The last of these assumes that $\mathbf{u}\neq\mathbf{0}$, so that projection perpendicular to $\mathbf{u}$ makes sense.) These should be obvious geometrically; in particular, the last of these states that the area of a parallelogram is the same as the area of a rectangle with the same base and height.

The cross product in three dimensions

For vectors in $\mathbf{R}^3$, we can interpret $\mathbf{u} \times \mathbf{v}$ as a vector. The magnitude $\|\mathbf{u} \times \mathbf{v}\|$ is the area from the previous section, so we only need to describe the direction of $\mathbf{u} \times \mathbf{v}$: it will be perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Most of the time, there are precisely two directions perpendicular to two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$. To decide which of these is the direction of $\mathbf{u} \times \mathbf{v}$, we use the *right-hand rule*: if you start by pointing the fingers of your right hand in the direction of $\mathbf{u}$, curl them to point in the direction of $\mathbf{v}$, and then stick out your thumb, then your thumb will point roughly in the direction of $\mathbf{u} \times \mathbf{v}$. (This should be used together with a right-handed coordinate system: if you point your fingers along the positive x-axis, curl them to point along the positive y-axis, and then stick out your thumb, then your thumb will point roughly along the positive z-axis.) If $\mathbf{u}$ and $\mathbf{v}$ happen to be parallel (or antiparallel), or if either (or both) of them is the zero vector $\mathbf{0}$, then this won't work; however, in that case, $\|\mathbf{u} \times \mathbf{v}\| = 0$, so then $\mathbf{u} \times \mathbf{v}$ must be $\mathbf{0}$, which has no direction.

Like the dot product, this operation distributes over addition and associates with scalar multiplication:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w},$$
$$\mathbf{u} \times a\mathbf{v} = a(\mathbf{u} \times \mathbf{v}).$$

The latter fact is easy to see, since we have a corresponding fact for $\|\mathbf{u} \times a\mathbf{v}\|$ and the direction of $\mathbf{u} \times a\mathbf{v}$ reverses when a is negative. The first of these is more difficult; it uses the result for $\|\mathbf{u} \times \mathbf{v}\|$ in terms of $\operatorname{proj}_{\mathbf{u}}^{\perp} \mathbf{v}$. This allows you to draw everything in the plane perpendicular to $\mathbf{u}$; if you look in the direction of $\mathbf{u}$ when looking at this plane, then $\mathbf{u} \times \mathbf{v}$ rotates $\operatorname{proj}_{\mathbf{u}}^{\perp} \mathbf{v}$ (which is in this plane) clockwise through a right angle and scales it by $\|\mathbf{v}\|$; since both this operation and projection distribute over addition, so does the cross product itself.

However, there is one important difference between the properties of the dot and cross products:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

This is because, while the magnitudes are the same, the directions are reversed, since you're curling your fingers the other way.

For practical calculations, it's again enough to know what happens to the standard basis vectors:

$$\mathbf{i} \times \mathbf{i} = \mathbf{0},\ \mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{i} \times \mathbf{k} = -\mathbf{j},$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k},\ \mathbf{j} \times \mathbf{j} = \mathbf{0},\ \mathbf{j} \times \mathbf{k} = \mathbf{i},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j},\ \mathbf{k} \times \mathbf{j} = -\mathbf{i},\ \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

Based on this,

$$\begin{aligned}\langle a, b, c\rangle \times \langle d, e, f\rangle &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = (bf - ce)\mathbf{i} + (cd - af)\mathbf{j} + (ae - bd)\mathbf{k} \\ &= \langle bf - ce, cd - af, ae - bd\rangle.\end{aligned}$$

For example,

$$\langle 1, -2, 0\rangle \times \langle 2, 2, 1\rangle = \langle (-2)(1) - (0)(2), (0)(2) - (1)(1), (1)(2) - (-2)(2)\rangle = \langle -2 - 0, 0 - 1, 2 + 4\rangle = \langle -2, -1, 6\rangle.$$

If you know about determinants, then you can think of

$$\langle a, b, c\rangle \times \langle d, e, f\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix};$$

the value of this determinant is the value of the cross product above.

Along with the cross product, people often look at the so-called *triple scalar product* of three vectors in $\mathbf{R}^3$; this is simply

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

This can be calculated with determinants as well; then the top row of the determinant, instead of consisting of the standard basis vectors, it now consists of the components of $\mathbf{u}$ to go with the components of $\mathbf{v}$ and $\mathbf{w}$ in the other rows. Geometrically, this represents a volume; more precisely, $|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}|$ is the volume of a parallelepiped whose edges are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ is positive if you can curl the fingers of your right hand from $\mathbf{u}$ to $\mathbf{v}$ and stick out your thumb along $\mathbf{w}$ but negative if your thumb points the wrong way.

The cross product in two dimensions

For vectors in $\mathbf{R}^2$, we can interpret $\mathbf{u} \times \mathbf{v}$ as a scalar, so this is sometimes callled the *scalar cross product*. The absolute value $|\mathbf{u} \times \mathbf{v}|$ is the $\|\mathbf{u} \times \mathbf{v}\|$ from above; $\mathbf{u} \times \mathbf{v}$ itself is positive if you turn counterclockwise to go from $\mathbf{u}$ to $\mathbf{v}$ but negative if you turn clockwise. (Here I'm assuming a counterclockwise coordinate system: the rotation from the positive x-axis to the positive y-axis is counterclockwise.) If $\mathbf{u}$ and $\mathbf{v}$ are parallel (or antiparallel), or if either of them is the zero vector $\mathbf{0}$, then $\mathbf{u} \times \mathbf{v}$ is just 0.

The cross product in 2 dimensions follows the same algebraic rules as in 3 dimensions:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w},$$
$$\mathbf{u} \times a\mathbf{v} = a(\mathbf{u} \times \mathbf{v}),$$
$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

If anything, these are easier to establish geometrically than the corresponding properties in $\mathbf{R}^3$.

Another way to think of the scalar cross product is to embed $\mathbf{R}^2$ within $\mathbf{R}^3$; that is, we take the z-coordinate of every point to be fixed (typically $z = 0$), so that the z-component of every vector is $\Delta z = 0$. Then instead of the scalar cross product $\mathbf{u} \times \mathbf{v}$, you can speak of the triple scalar product $\mathbf{k} \cdot \mathbf{u} \times \mathbf{v}$. Yet another way to think of it is as a dot product; much as $a - b$ is the sum of a and $-b$, so $\mathbf{u} \times \mathbf{v}$ is the dot product of $\mathbf{u}$ and $\times\mathbf{v}$, where $\times\mathbf{v}$ is simply $\mathbf{v}$ rotated clockwise through a right angle. You can also speak of signed angles in 2 dimensions; if you treat a counterclockwise angle as positive and a clockwise angle as negative, then

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \bar{\angle}(\mathbf{u}, \mathbf{v}),$$

where the bar over the angle symbol indicates a signed angle.

For practical calculations, since $\mathbf{i} \times \mathbf{i} = 0$, $\mathbf{i} \times \mathbf{j} = 1$, $\mathbf{j} \times \mathbf{i} = -1$, and $\mathbf{j} \times \mathbf{j} = 0$, the formula is

$$\langle a, b \rangle \times \langle c, d \rangle = ad - bc.$$

For example,

$$\langle 1, -2 \rangle \times \langle 3, 5 \rangle = (1)(5) - (-2)(3) = 5 + 6 = 11.$$

If you know about determinants, then

$$\langle a, b \rangle \times \langle c, d \rangle = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Similarly,

$$\times\langle a, b \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ a & b \end{vmatrix} = \langle b, -a \rangle.$$

Cross products in more than 3 dimensions can also be done, but in that case the result is neither a scalar nor a vector but a more general concept called a *tensor*. We will not be using these.

Orientation

The dot and cross products both rely on the geometric notion of length, but the cross product additionally depends on an **orientation**; this is the choice between the right-hand and left-hand rules (in 3 dimensions) or between counterclockwise or clockwise angles (in 2 dimensions). While our physical space really does have lengths and angles, the choice of orientation is arbitrary, so results that apply to geometry shouldn't depend on it.

Just as we can distinguish row vectors from column vectors in situations where lengths don't make sense, so we can distinguish axial vectors from polar vectors in situations where orientation is arbitrary. So, a **polar vector** is an ordinary vector representing a change in position, but an **axial vector** or **pseudovector** is a vector together with a choice of orientation, where we may reverse our choice of orientation as we please so long as we replace the vector with its opposite when we do so. For example, while a polar vector in $\mathbf{R}^3$ may be fully described as $\langle -2, -1, 6\rangle$, an axial vector in $\mathbf{R}^3$ might be described as $\langle -2, -1, 6\rangle$ right-handed, or (for the *same* axial vector) as $\langle 2, 1, -6\rangle$ left-handed. Thus you can say, for example,

$$\langle 1, -2, 0\rangle \times \langle 2, 2, 1\rangle = \langle -2, -1, 6\rangle \text{ right-handed} = \langle 2, 1, -6\rangle \text{ left-handed.}$$

Similarly, a **pseudoscalar** is a scalar together with a choice of orientation, where again we may reverse our choice of orientation as we please so long as we replace the scalar with its opposite. In $\mathbf{R}^2$, the cross product of two vectors is a pseudoscalar; in $\mathbf{R}^3$, the triple scalar product of three vectors is a pseudoscalar. For example,

$$\langle 1, -2\rangle \times \langle 3, 5\rangle = 11 \text{ counterclockwise} = -11 \text{ clockwise.}$$

Axial vectors obey the same rules of arithmetic as polar vectors; here is a list of operations with these that make sense in $\mathbf{R}^3$:

- Addition: adding a polar vector to a point to get another point, adding two polar vectors together to get another polar vector, adding two axial vectors together to get another axial vector;
- Subtraction: subtracting a polar vector from a point to get another point, subtracting one polar vector from another to get another polar vector, subtracting one axial vector from another to get another axial vector;
- Scalar multiplication: multiplying a polar vector by a scalar to get another polar vector, multiplying an axial vector by a scalar to get another axial vector, multiplying a polar vector by a pseudoscalar to get an axial vector, multiplying an axial vector by a pseudoscalar to get a polar vector;
- Inner multiplication (dot product): multiplying two polar vectors to get a scalar, multiplying a polar vector and an axial vector to get a pseudoscalar, multiplying two axial vectors to get a scalar;
- Outer multiplication (cross product): multiplying two polar vectors to get an axial vector, multiplying a polar vector and an axial vector to get a polar vector, multiplying two axial vectors to get an axial vector.

Similarly, pseudoscalars can be added or subtracted to produce more pseudoscalars and can be multiplied together to produce an ordinary scalar, or you can multiply a scalar and a pseudoscalar to produce another pseudoscalar. In $\mathbf{R}^2$, the list of operations is the same, except that the result of a cross product is a scalar or a pseudoscalar rather than a vector (a polar vector) or a pseudovector (an axial vector).

The rule of thumb for all of this is that you can only add or subtract things that are alike in every way, but you can multiply anything together; the result is 'pseudo' if you multiplied together an odd number of pseudothings (so pseudos cancel, like minus signs, in pairs), where the cross product introduces an extra pseudo.

In the most general case, where you don't have a good notion of length and also don't have any way to prefer one orientation over another, you have polar column vectors (the ordinary notion of vector), axial column vectors, polar row vectors, and axial row vectors. In general, only polar column vectors can interact with points. None of this affects calculations when properly done, but like keeping track of units, keeping track of these can prevent you from accidentally doing meaningless calculations.