Vectors

In this class, we look at spaces with up to 3 dimensions, but most of the ideas in this course (and the next) continue to make sense in spaces with any whole number of dimensions. Although spaces with more than 3 dimensions are difficult to visualize, since we're used to living in a 3-dimensional space, they make perfect sense mathematically. Furthermore, whenever you're trying to keep track of 4 or more independent quantities at once, then you need the mathematics of a space with 4 or more dimensions, whether or not you choose to visualize that space geometrically.

If we assign rectangular coordinates to a space of n dimensions, then the result is called $\mathbf{R}^n$ (or $\mathbb{R}^n$); in particular, a coordinate space of 1 dimension is $\mathbf{R}^1$ or simply $\mathbf{R}$, which is the set of real numbers, or (thinking geometrically) the real number line. You can call the coordinates whatever you like, but it's most common to use x (or sometimes t) as the coordinate in $\mathbf{R}$; then to use x and y as the coordinates in $\mathbf{R}^2$; then x, y, and z in $\mathbf{R}^3$; and finally x_1, x_2, $\ldots$, and x_n in $\mathbf{R}^n$ generally. But there are other systems; as long as you list n independent variables in a row, then you have a valid list of coordinates for $\mathbf{R}^n$.

A **point** in $\mathbf{R}^n$ may be denoted by listing the values of its coordinates in order, separated by commas and optionally surrounded by grouping parentheses. Thus, (x) or (more commonly) x gives a point in the real line $\mathbf{R}$, while (x, y) gives a point in the coordinate plane $\mathbf{R}^2$, (x, y, z) gives a point in the coordinate space $\mathbf{R}^3$, and $(x_1, x_2, \ldots, x_n)$ gives a point in $\mathbf{R}^n$ (which is the most general case).

Sometimes it's nice to have a way to refer to a point in any number of dimensions without having to write a long list with dots in it; then I usually write P for the point. Thus, in 1 dimension, $P = x$; in 2 dimensions, $P = (x, y)$; in 3 dimensions, $P = (x, y, z)$; and in n dimensions, $P = (x_1, x_2, \ldots, x_n)$. So for example, if I say that $P = (2, 3, 5)$, then this is the same as saying that $x = 2$, $y = 3$, and $z = 5$.

It's traditional to use uppercase letters to name points, as I just did. Another tradition is to leave out the equality sign when naming points; so instead of writing $P = (2, 3, 5)$ as I did above, people often just write $P(2, 3, 5)$. I think that this is a terrible convention, so I won't follow it, but you will see it sometimes, even in the textbook.

Vectors

A **vector** is a movement between points. For example, to move in the plane from the point $(2, 3)$ to the point $(3, 1)$, you move 1 unit to the right (in the positive x direction) and 2 units downwards (in the negative y direction). This movement —1 unit to the right and 2 units downwards— is a vector.

A vector in $\mathbf{R}^n$ has the same amount of information as a point there: n real numbers. For this reason, people sometimes write a vector using the same notation as they use to write a point. For example, the vector from the previous paragraph could be written as $(1, -2)$, the same notation as used for the point $(1, -2)$. When referring to a vector, $(1, -2)$ means a movement 1 unit to the right and 2 units downwards; when referring to a point, $(1, -2)$ means the point that lies 1 unit to the right and 2 units downwards from the origin.

However, a vector is not the same thing as a point, and so people often use different notation instead. Common notations for the vector that I've been talking about include $[1, -2]$ and $\langle 1, -2 \rangle$. I will use the last of these, since that is used in the textbook. (There is another notation, which the book uses even more often than $\langle 1, -2 \rangle$, and that is $\mathbf{i} - 2\mathbf{j}$. However, I'll save that for later.) The terminology for these numbers is also different; while 1 and -2 are the *coordinates* of the point $(1, -2)$, we say that 1 and -2 are the **components** of the vector $\langle 1, -2 \rangle$.

Whereas a point tells you a location, a vector tells you only about the motion and nothing about the location. So the vector from $(2, 3)$ to $(3, 1)$ is the same vector as, say, the vector from $(-2, 7)$ to $(-1, 5)$. In both cases, the motion is 1 unit to the right and 2 units downwards, so the vector is $\langle 1, -2 \rangle$.

Motion on a number line corresponds arithmetically to addition. For example, if you start at the number 2 on a number line and move 4 units to the right, then you end up at the number 6, and we represent this fact in arithmetic as $2 + 4 = 6$. Similarly, if you start at $(2, 3)$ and move according to the vector $\langle 1, -2 \rangle$, then you end up at $(3, 1)$, and we represent this fact in arithmetic as $(2, 3) + \langle 1, -2 \rangle = (3, 1)$. So you can add a point and a vector to get another point. Or from another perspective, we could write $6 - 2 = 4$, and similarly $(3, 1) - (2, 3) = \langle 1, -2 \rangle$. So one way to describe a vector is to say that it's what you get when you subtract two points. The textbook doesn't talk about arithmetic with points and vectors like this; it does talk about calculating the vector from one point to another or calculating the point reached from another point by following a given vector, but it doesn't refer to these operations as subtraction and addition. Nonetheless, that's exactly what they are.

The rules for these calculations are very straightforward: you add or subtract corresponding coordinates and components. That is, to get the first coordinate of the sum, you add the first coordinate of the original point and the first component of the vector, and similarly for the second coordinate; or when you subtract two vectors, you subtract the first coordinates of the two points to get the first component of the difference, and similarly for the second component. So you can write out the calculations in full thus:

$$(2,3) + \langle 1,-2 \rangle = (2+1, 3-2) = (3,1);$$
$$(3,1) - (2,3) = \langle 3-2, 1-3 \rangle = \langle 1,-2 \rangle.$$

Here are general formulas for this rule in any number of dimensions:

$$(a_1, a_2, \ldots, a_n) + \langle v_1, v_2, \ldots, v_n \rangle = (a_1 + v_1, a_2 + v_2, \ldots, a_n + v_n);$$
$$(b_1, b_2, \ldots, b_n) - (a_1, a_2, \ldots, a_n) = \langle b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n \rangle.$$

When I use P to denote a generic point, I'll use ΔP to denote a generic vector. Here, the uppercase Greek letter Delta, 'Δ', which stands for 'difference', is commonly used to indicate the amount by which the value of some quantity changes. (Think of $\Delta y/\Delta x$ for the slope of a line.) That is,

$$\Delta P = P_1 - P_0,$$

or

$$\Delta P = \langle \Delta x_1, \Delta x_2, \ldots, \Delta x_n \rangle.$$

When you give a vector a name of its own, however, it's common to use a boldface lowercase letter, such as $\mathbf{u}$ or $\mathbf{v}$. Thus, if I use $\mathbf{v}$ to refer to the vector that I've been using as an example throughout this section, then I would write $\mathbf{v} = \langle 1,-2 \rangle$. In handwriting, you can write a little arrow over the letter instead, to produce something like $\vec{v}$; other common conventions are to underline or overline vectors, producing symbols such as $\underline{v}$ or $\overline{v}$. On the other hand, it's OK to just write v if you want. The meaning of any symbol that you use should be clear from the context that you provide; in particular, the context should make clear whether a symbol refers to a number, function, point, vector, or whatever, regardless of whatever fancy fonts or decorations you may or may not use.

Arithmetic with vectors

Besides adding vectors to points and subtracting points to get a vector, you can also do arithmetic within the world of vectors itself. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in n dimensions, both representing some motion within $\mathbf{R}^n$, then $\mathbf{u} + \mathbf{v}$ represents the motion of $\mathbf{u}$ followed by the motion of $\mathbf{v}$. This is consistent with how addition of motions works on a number line; for example, if you move 4 units to the right and then move 3 units to the right, then overall you're moving $4 + 3 = 7$ units to the right.

If $\mathbf{v}$ is a vector, then $-\mathbf{v}$ is the vector representing the opposite motion. Again, this matches arithmetic on a number line; the opposite of moving 4 units to the right is moving 4 units to the left, which is represented by the number -4. Then $\mathbf{u} - \mathbf{v}$ just means $\mathbf{u} + (-\mathbf{v})$.

You calculate these by the same principles as arithmetic between points and vectors. For example, to add $\langle 1,-2 \rangle$ and $\langle 3,5 \rangle$, you simply add the corresponding components:

$$\langle 1,-2 \rangle + \langle 3,5 \rangle = \langle 1+3, -2+5 \rangle = \langle 4,3 \rangle.$$

And this should make sense; if you move 1 unit to the right and 2 units downwards, then move 3 units to the right and 5 units upwards, then overall you're moving 4 units to the right and 3 units upwards. Similarly,

$$\langle 1,-2 \rangle - \langle 3,5 \rangle = \langle 1-3, -2-5 \rangle = \langle -2,-7 \rangle.$$

That is, if you move 1 unit to the right and 2 units downwards and then move the opposite of 3 units to the right and 5 units upwards (which is 3 units to the left and 5 units downwards), then overall you're moving 2 units to the left and 7 units downwards. Here are the general formulas in $\mathbf{R}^n$:

$$\langle u_1, u_2, \ldots, u_n \rangle + \langle v_1, v_2, \ldots, v_n \rangle = \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle;$$
$$\langle u_1, u_2, \ldots, u_n \rangle - \langle v_1, v_2, \ldots, v_n \rangle = \langle u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n \rangle.$$

Besides adding and subtracting vectors, you can multiply or divide them by real numbers. For example, if $\mathbf{v}$ is a vector representing some motion, then $2\mathbf{v}$ represents doing that motion twice, $1/2\,\mathbf{v}$ or $\mathbf{v}/2$ represents performing half of that motion, $-2\mathbf{v}$ represents making the opposite motion twice, and so on. You calculate these by multiplying each component by that same real number; for example,

$$2\langle 1, -2 \rangle = \langle 2(1), 2(-2) \rangle = \langle 2, -4 \rangle,$$
$$\frac{1}{2}\langle 1, -2 \rangle = \left\langle \frac{1}{2}(1), \frac{1}{2}(-2) \right\rangle = \left\langle \frac{1}{2}, -1 \right\rangle \text{ or}$$
$$\frac{\langle 1, -2 \rangle}{2} = \left\langle \frac{1}{2}, \frac{-2}{2} \right\rangle = \left\langle \frac{1}{2}, -1 \right\rangle, \text{ and}$$
$$-2\langle 1, -2 \rangle = \langle -2(1), -2(-2) \rangle = \langle -2, 4 \rangle.$$

Here are the general formulas in $\mathbf{R}^n$:

$$a\langle v_1, v_2, \ldots, v_n \rangle = \langle av_1, av_2, \ldots, av_n \rangle;$$
$$\frac{\langle v_1, v_2, \ldots, v_n \rangle}{a} = \left\langle \frac{v_1}{a}, \frac{v_2}{a}, \ldots, \frac{v_n}{a} \right\rangle \text{ for } a \neq 0.$$

This operation is called **scalar multiplication** (or *scalar division*), because geometrically it amounts to changing the scale used to measure the vector (at least when the real number in question is positive). As a result of this, numbers are often called **scalars** when working with vectors, even though the word 'number' would work perfectly well.

More generally, you can take any homogeneous linear expression (that is a linear expression without a constant term) in any number of variables, replace the variables with vectors, and get a legitimate operation on vectors. Such an operation is called, in general, a **linear combination**. For example, $2\mathbf{u} + 3\mathbf{v} - 5\mathbf{w}$ is a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Geometrically, this represents moving twice according to $\mathbf{u}$, then moving 3 times according to $\mathbf{v}$, and moving 5 times the reverse of the motion given by $\mathbf{w}$.

Still more generally, you can replace the variables with points or vectors; if the sum of the coefficients on the points is 0, then the result is a vector, and if the sum of the coefficients on the points is 1, then the result is a point. For example, if A, B, and C are points, while $\mathbf{u}$ and $\mathbf{v}$ are vectors, then $2A - 3B + 2C + 4\mathbf{u} - 5\mathbf{v}$ is a point (because $2 - 3 + 2 = 1$), while $2A - 3B + C + 4\mathbf{u} - 5\mathbf{v}$ is a vector (because $2 - 3 + 1 = 0$). Geometrically, $2A - 3B + 2C + 4\mathbf{u} - 5\mathbf{v}$ means the point that you reach by starting at A, moving as you would move to get to A from B, then moving twice as you would move to get to C from B, then moving 4 times according to $\mathbf{u}$, and moving 5 times the reverse of the motion given by $\mathbf{v}$. (That is, think of it as $A + (A - B) + 2(C - B) + 4\mathbf{u} - 5\mathbf{v}$.) Similarly, $2A - 3B + C + 4\mathbf{u} - 5\mathbf{v}$ is the motion consisting of moving twice as you would move to get to A from B, then moving as you would move to get to C from B, then moving 4 times according to $\mathbf{u}$, and moving 5 times the reverse of the motion given by $\mathbf{v}$. (That is, think of it as $2(A - B) + (C - B) + 4\mathbf{u} - 5\mathbf{v}$.)

Another example of a point is $1/3\,A + 1/3\,B + 1/3\,C$, which is the average of the 3 points. If you think of this as $A + 2/3\,(B - A) + 1/3\,(C - B)$, then you can describe this in terms similar to those of the previous examples, but in this case it's probably better to think of it directly as an average.

If the sum of the coefficients on the points is neither 1 nor 0, then there is no direct geometric interpretation of the linear combination, but you can still perform calculations with such things; they basically represent internal parts of a larger calculation, such as the $2A - 3B$ that begins some of the examples above.

All of the usual algebraic identities apply to linear combinations of points and vectors. For example, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$, $(A+\mathbf{u})+\mathbf{v}=A+(\mathbf{u}+\mathbf{v})$, $2(\mathbf{u}+\mathbf{v})=2\mathbf{u}+2\mathbf{v}$, and so on. Although you can prove these geometrically, the simplest way to verify them is to do so component by component; then they reduce to identities about real numbers.

You could try multiplying and dividing vectors by each other using the same method of calculation as you use for adding and subtracting them, component by component. People do this sometimes, but there's no geometric interpretation of these operations, neither directly nor as part of a larger calculation with a geometric interpretation. So we won't be doing that. Instead, we'll see some other methods of multiplying vectors later on.

The **zero vector**, denoted $\mathbf{0}$, represents no motion at all. It's general formula in $\mathbf{R}^n$ is

$$\mathbf{0}=\langle 0,0,\ldots,0\rangle.$$

It obeys algebraic rules analogous to those obeyed by the real number 0, such as $\mathbf{0}+\mathbf{v}=\mathbf{v}$, $\mathbf{v}-\mathbf{v}=\mathbf{0}$, and $A+\mathbf{0}=A$. (The last of these demonstrates what it means to say that $\mathbf{0}$ represents no motion at all; you start at the point A, do nothing, and wind up still at A.)

The standard basis vectors

There are some other special symbols for special vectors, and these lead to another general system of notation for vectors (and points).

In $\mathbf{R}^2$, there are 2 **standard basis vectors**, $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{i}=\langle 1,0\rangle,\ \mathbf{j}=\langle 0,1\rangle.$$

In $\mathbf{R}^3$, there are 3 of them:

$$\mathbf{i}=\langle 1,0,0\rangle,\ \mathbf{j}=\langle 0,1,0\rangle,\ \mathbf{k}=\langle 0,0,1\rangle.$$

In $\mathbf{R}^n$, there is a shift in the usual notation:

$$\mathbf{e}_1=\langle 1,0,0,\ldots,0\rangle,\ \mathbf{e}_2=\langle 0,1,0,0,\ldots,0\rangle,\ \ldots,\ \mathbf{e}_n=\langle 0,0,\ldots,0,0,1\rangle.$$

The value of this is that any vector can be written as a unique linear combination of the standard basis vectors:

$$\langle a,b\rangle=a\mathbf{i}+b\mathbf{j};$$
$$\langle a,b,c\rangle=a\mathbf{i}+b\mathbf{j}+c\mathbf{k};$$
$$\langle a_1,a_2,\ldots,a_n\rangle=a_1\mathbf{e}_1+a_2\mathbf{e}_2+\cdots+a_n\mathbf{e}_n.$$

Work out the right-hand sides of these and see for yourself that you get the left-hand side. (It's a little annoying that $\mathbf{i}$ and $\mathbf{j}$ are ambiguous, but as long as you know whether they're supposed to be in $\mathbf{R}^2$ or in $\mathbf{R}^3$, then you know what they mean.)

If a component of a vector happens to be 1, then you can leave it out of the expression in the standard basis vectors; if the component is negative, then you use subtraction instead of addition; if the component is 0, then you leave that term out entirely. For example, $\langle 1,-2\rangle=1\mathbf{i}+(-2)\mathbf{j}=\mathbf{i}-2\mathbf{j}$. In $\mathbf{R}^3$, $\langle 1,-2,0\rangle$ is also written $\mathbf{i}-2\mathbf{j}$, because the component on $\mathbf{k}$ is 0.

You can now do arithmetic with vectors by following the ordinary rules of algebra and leaving the symbols for the standard basis vectors alone. For example, instead of $\langle 1,-2\rangle+\langle 3,5\rangle=\langle 4,3\rangle$, you calculate

$$(\mathbf{i}-2\mathbf{j})+(3\mathbf{i}+5\mathbf{j})=(1+3)\mathbf{i}+(-2+5)\mathbf{j}=4\mathbf{i}+3\mathbf{j}.$$

Similarly, instead of $2\langle 1,-2\rangle=\langle 2,-4\rangle$, you calculate

$$2(\mathbf{i}-2\mathbf{j})=2\mathbf{i}-2(2\mathbf{j})=2\mathbf{i}-4\mathbf{j}.$$

You can even extend this notation to points by introducing O for the origin of the coordinate system. That is,

$$O=(0,0,\ldots,0)$$

in $\mathbf{R}^n$. Then any point can be described by starting at the origin and moving along a vector whose components are the coordinates of that point; for example, $(2,3) = O + \langle 2,3\rangle = O + 2\mathbf{i} + 3\mathbf{j}$. Then you can again do calculations using the rules of algebra; for example, instead of $(2,3) + \langle 1,-2\rangle = (3,1)$, you calculate

$$(O + 2\mathbf{i} + 3\mathbf{j}) + (\mathbf{i} - 2\mathbf{j}) = O + (2+1)\mathbf{i} + (3-2)\mathbf{j} = O + 3\mathbf{i} + \mathbf{j}.$$

The textbook uses this notation for vectors most of the time, although it continues to use a list of coordinates with commas for points, which it has to do since it never refers directly to addition of points and vectors.

Lengths and angles

In many situations, we want to refer to the distance between two points, or equivalently to the length of a vector. This goes by several names; in general, the **length**, **magnitude**, or **norm** of a vector in $\mathbf{R}^n$ is

$$\|\langle v_1, v_2, \ldots, v_n\rangle\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}.$$

Here I've denoted the length of a vector $\mathbf{v}$ as $\|\mathbf{v}\|$, although the textbook writes this as simply $|\mathbf{v}|$ instead. As a statement about distances, this is the n-dimensional generalization of the Pythagorean Theorem.

One basic algebraic property of lengths is

$$\|a\mathbf{v}\| = |a|\,\|\mathbf{v}\|.$$

(Note that you must write $|a|$ when a is a scalar, even if you choose to use the notation $\|\mathbf{v}\|$ when $\mathbf{v}$ is a vector.) You can check this from the general formula by factoring inside the square root; remember the identity $\sqrt{a^2} = |a|$ for arbitrary real numbers. (It's a common algebra mistake to think that $\sqrt{a^2} = a$; this is correct when $a \geq 0$ but not otherwise.) In particular,

$$\|-\mathbf{v}\| = \|\mathbf{v}\|.$$

Also,

$$\|\mathbf{0}\| = 0;$$

conversely, if $\|\mathbf{v}\| = 0$, then it must be that $\mathbf{v} = \mathbf{0}$. (Ultimately this is because a sum of squares of real numbers can only be zero if all of the original numbers are zero.)

There is no general formula for $\|\mathbf{u} + \mathbf{v}\|$; however, we can say

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

This is called the **triangle inequality**, since if you draw a triangle whose sides are $\mathbf{u}$, $\mathbf{v}$, and their sum $\mathbf{u} + \mathbf{v}$, then this expresses the fact that the length of the last side is the shortest distance between its two endpoints. You can check this from the formula by squaring both sides, cancelling some common terms, squaring again, subtracting the two sides, and factoring the result as a perfect square. You can then argue that this perfect square is greater than or equal to zero, so the right-hand side just before the subtraction is greater than or equal to the left-hand side at that stage, and this remains so upon taking principal square roots, adding some common terms, and taking principal square roots again. I'll skip the details.

If $\mathbf{v} \neq \mathbf{0}$ (so that you can divide by $\|\mathbf{v}\|$), then $\mathbf{v}/\|\mathbf{v}\|$ is a vector whose own magnitude is 1. (This is because

$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \frac{\|\mathbf{v}\|}{|\|\mathbf{v}\||} = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1,$$

using that $\|\mathbf{v}\| \geq 0$.) This is called the **unit vector** in the direction of $\mathbf{v}$, or simply the **direction** of $\mathbf{v}$. The usual notation for this is $\hat{\mathbf{v}}$:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

For some reason, the book never introduces this notation (or any other notation for this concept), but it refers to the idea itself quite often. Notice that you can write $\mathbf{v} = \|\mathbf{v}\|\hat{\mathbf{v}}$; this expresses the common slogan that a vector has both a length and a direction. (However, the zero vector has only a length, of 0, and no way to pick out any unit vector as its direction.)

If you perform some algebraic tricks with the triangle inequality and assume that neither $\mathbf{u}$ nor $\mathbf{v}$ is the zero vector $\mathbf{0}$ (so that you can divide by their norms), then you can also derive the compound inequality

$$-1 \leq \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{2\,\|\mathbf{u}\|\,\|\mathbf{v}\|} \leq 1.$$

(I'll skip this derivation, but it's based on first replacing $\mathbf{v}$ with $-\mathbf{v}$, squaring both sides, and rearranging terms to derive one half of this result, then going back to the beginning and replacing $\mathbf{u}$ with $\mathbf{u} - \mathbf{v}$, squaring both sides again, and rearranging terms to derive the other half of the result.) If you draw a triangle whose sides are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ (so that $\mathbf{u}$ and $\mathbf{v}$ are both starting from the same point), then the Law of Cosines says that the expression in the middle of the compound inequality above is the cosine of the angle between the sides $\mathbf{u}$ and $\mathbf{v}$, and the inequality verifies that this lies within the possible range of values for a cosine. (If either $\mathbf{u}$ or $\mathbf{v}$ is the zero vector, then you don't really have a triangle, and this angle doesn't make sense.)

If you have two rays emanating from the same point in a multidimensional space, then the only way to describe the angle between them is with an angle between 0 and π (or 180°), which is the range of possible values of an arccosine (or inverse cosine), so taking the arccosine of the expression above gives you this angle:

$$\angle(\mathbf{u}, \mathbf{v}) = \arccos\left(\frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{2\,\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

(In $\mathbf{R}^2$, and only in $\mathbf{R}^2$, it's possible to distinguish clockwise and counterclockwise angles, which I'll come back to when I discuss the cross product below.) Thus, it's possible to describe both lengths and angles using vectors, through the concept of the magnitude of a vector. (There's a more efficient way to calculate this cosine, which we'll see later on using the dot product, but it's important that angles can be calculated from lengths alone.)

However, for many applications of vectors, the concept of length or magnitude really doesn't make sense! This is because vectors describe motion within any space with any coordinates, and those coordinates might refer to incompatible quantities. For example, if x measures time and y measures something that changes with time but is not itself a time (the height of a falling object, the price of a stock, the population of the world, or nearly any other quantity of interest), then it really doesn't make sense to talk about the magnitude

$$\|\Delta P\| = \|\langle \Delta x, \Delta y \rangle\| = \sqrt{\Delta x^2 + \Delta y^2}.$$

You can see this if you imagine what units of measurement you might use for such a magnitude; if x is measured in seconds and y is measured in metres (as one might do when talking about the height of a falling object, for example), then which unit is $\|\Delta P\|$ in? Neither one makes sense, nor does any combination of them.

So while lengths of vectors and angles between them always exist in the realm of mathematical abstraction, they can only be meaningful when all of the coordinates measure the same type of quantity. (Even then, these concepts may or may not really be meaningful, but at least they have a chance.)