Parametrized curves

An important application of vectors is to Calculus on parametrized curves.

Point- and vector-valued functions

Besides individual points and vectors, one can also consider variable points and vectors, which are the outputs of point- and vector-valued functions. A **point-valued function** in $\mathbf{R}^n$ consists of n ordinary functions, all with the same domain. For example, a point-valued function in $\mathbf{R}^2$ consists of 2 functions with the same domain, say $f(t) = t^2$ and $g(t) = t^3$. We put these together into a single function (f, g), which takes a real-number t as input and produces the point

$$(f, g)(t) = \big(f(t), g(t)\big) = (t^2, t^3) = \mathrm{O} + t^2\mathbf{i} + t^3\mathbf{j}$$

(in this case) as output. A **vector-valued function** in $\mathbf{R}^n$ also consists of n ordinary functions, all with the same domain. But now we think of the output as a vector:

$$\langle f, g\rangle(t) = \langle f(t), g(t)\rangle = \langle t^2, t^3\rangle = t^2\mathbf{i} + t^3\mathbf{j}$$

(for example). If we want to know whether one of these functions is continuous or differentiable, then we just look at each of its components separately. For example, since the functions f and g above are continuous and differentiable everywhere, so are (f, g) and $\langle f, g\rangle$.

The textbook often doesn't distinguish between a point P and its position vector $\mathbf{r} = P - \mathrm{O}$, where O is the origin of a coordinate system. Conceptually, they're very different, since you can talk about points and vectors geometrically without bringing coordinates into it, so the concepts are meaningful even if there is no inherent point O to equivocate them. On the other hand, when doing calculations, it's easy to conflate them; since the coordinates of O are all 0, when you do the subtraction $P - \mathrm{O}$ to get $\mathbf{r}$, you find that the components of $\mathbf{r}$ are exactly the same as the coordinates of P. Still, you should always keep in mind whether a given expression really refers to a point or to a vector.

In particular, a point-valued function can be viewed as a **parametrized curve**; each value of the input t (which in this context is called a *parameter*) gives a point, and all of these points together make up a curve. A vector-valued function only defines a curve by interpreting each vector with reference to point O deemed to be the origin, but that is how the textbook insists on doing it starting in Chapter 12. (You'll see this if you take Calculus 3, but it's not an issue yet in Chapter 10, since the textbook isn't discussing vectors there.)

If P is a point, then the difference ΔP is a vector (because it's the result of subtracting two points), and then the differential $\mathrm{d}P$ is an infinitesimal vector. If P is a function of some scalar quantity t, then $\mathrm{d}P/\mathrm{d}t$ makes sense, because it's a vector divided by a scalar, but now it's no longer infinitesimal (unless it happens to be zero). In other words, *the derivative of a point with respect to a scalar is a vector.* Another way to see this is that if F is a point-valued function, then its derivative F' is a vector-valued function:

$$F'(t) = \lim_{h \to 0} \left(\frac{F(t+h) - F(t)}{h}\right);$$

first subtract two points to get a vector, then divide by the scalar h to get another vector, and finally take the limit of these vectors to get a vector. Of course, the derivative of a *vector* with respect to a scalar is *also* a vector; in other words, the derivative of a vector-valued function is also a vector-valued function.

For example, if P gives the position of some object at time t, then P is a point, but $\mathrm{d}P/\mathrm{d}t$, the *velocity* of the object, is a vector. (Note that the magnitude of this vector is the object's *speed*.) If we write $\mathbf{v}$ for $\mathrm{d}P/\mathrm{d}t$ (which can also be written as $\mathrm{d}\mathbf{r}/\mathrm{d}t$), then $\mathrm{d}\mathbf{v}/\mathrm{d}t$ is the acceleration of the object, which is also a vector. (Physicists and mechanical engineers use the word ‘acceleration’ like this, to indicate any change in velocity —speed or direction— over time. In everyday language, this word means something more like $\mathrm{d}\|\mathbf{v}\|/\mathrm{d}t$, the derivative of speed with respect to time, which is the same as the scalar component of the acceleration in the direction of the velocity. This is positive if the object is speeding up and negative if the object is slowing down, or decelerating. Section 12.5 of the textbook discusses all of this in detail.)

Reversing this, if you take the indefinite integral of a vector, then the result may be either a point *or* a vector, because differentiating either of these yields a vector. This ambiguity is packaged into the constant of integration. For example, $\int \langle 2t, 3 \rangle \, \mathrm{d}t = \langle t^2, 3t \rangle + C$, which is a point if C is a point and a vector if C is a vector. (If C is a vector, then you may want to call it $\mathbf{C}$ instead, but that is just a convention, not a requirement.) The definite integral of a vector, however, is always a vector: fundamentally, you get it by adding up infinitely many infinitesimal vectors (or approximate it by adding up a large number of small vectors), and adding up vectors yields a vector; in practice, you usually calculate it by subtracting indefinite integrals, and regardless of whether you view the indefinite integrals as points or as vectors, subtracting them yields a vector. For example, both $\int_{t=0}^{1} \langle 2t, 3 \rangle \, \mathrm{d}t = \langle t^2, 3t \rangle |_{t=0}^{1} = \langle 1, 3 \rangle - \langle 0, 0 \rangle = \langle 1, 3 \rangle$, and $\int_{t=0}^{1} \langle 2t, 3 \rangle \, \mathrm{d}t = (t^2, 3t)|_{t=0}^{1} = (1, 3) - (0, 0) = \langle 1, 3 \rangle$ give the same result. In fact, either of them could be packaged up as

$$\int_{t=0}^{1} \langle 2t, 3 \rangle \, \mathrm{d}t = \left\langle \int_{t=0}^{1} 2t \, \mathrm{d}t, \int_{t=0}^{1} 3 \, \mathrm{d}t \right\rangle = \left\langle t^2 |_{t=0}^{1}, 3t|_{t=0}^{1} \right\rangle = \langle 1 - 0, 3 - 0 \rangle = \langle 1, 3 \rangle.$$

Putting this all together, consider the initial-value problem in which the acceleration of an object is $-32\mathbf{k} = \langle 0, 0, -32 \rangle$ (which is the acceleration of a freely falling object near Earth's surface, if we use units of feet and seconds), the object's initial velocity is $\langle 3, 0, 4 \rangle$ (so a speed of $5\,\mathrm{ft/s}$ eastward and upward with a slope of $4/3$), and the object's initial position is $(0, 0, 100)$ (so 100 feet above the origin on the ground). Then you can calculate a general formula for the object's position P as a function of the elapsed time t by integrating:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \langle 0, 0, -32 \rangle; \\
\mathrm{d}\mathbf{v} &= \langle 0, 0, -32 \rangle \, \mathrm{d}t; \\
\int_{\mathbf{v}=\langle 3,0,4 \rangle} \mathrm{d}\mathbf{v} &= \int_{t=0} \langle 0, 0, -32 \rangle \, \mathrm{d}t; \\
\mathbf{v} - \langle 3, 0, 4 \rangle &= \langle 0, 0, -32t \rangle - \langle 0, 0, -32(0) \rangle; \\
\mathbf{v} &= \langle 3, 0, 4 \rangle + \langle 0, 0, -32t \rangle; \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \langle 3, 0, 4 - 32t \rangle; \\
\mathrm{d}P &= \langle 3, 0, 4 - 32t \rangle \, \mathrm{d}t; \\
\int_{P=(0,0,100)} \mathrm{d}P &= \int_{t=0} \langle 3, 0, 4 - 32t \rangle \, \mathrm{d}t; \\
P - (0, 0, 100) &= \langle 3t, 0, 4t - 16t^2 \rangle - \left\langle 3(0), 0, 4(0) - 16(0)^2 \right\rangle; \\
P &= (0, 0, 100) + \langle 3t, 0, 4t - 16t^2 \rangle; \\
P &= (3t, 0, 100 + 4t - 16t^2).
\end{aligned}$$

In other words, the position after t seconds is $3t$ feet east of the origin at a height of $100 + 4t - 16t^2$ feet.

(In the course of solving this, I've used the *semidefinite integral*:

$$\int_{t=a} f(t) \, \mathrm{d}t = \int_{\tau=a}^{t} f(\tau) \, \mathrm{d}\tau.$$

The Fundamental Theorem of Calculus allows us to calculate these integrals easily:

$$\int_{t=a} F'(t) \, \mathrm{d}t = F(t) - F(a).$$

This is very handy when solving initial-value problems. Since $\mathbf{v} = \langle 3, 0, 4 \rangle$ when $t = 0$, I was doing the same operation to both sides of the equation in the first step in which I introduced semidefinite integrals; similarly, the second introduction of semidefinite integrals is valid because $P = (0, 0, 100)$ when $t = 0$. To solve this problem using indefinite integrals instead requires two extra steps —one for each integration— to find the constants associated with the indefinite integrals, but using semidefinite integrals avoids that.)

Differentiation of parametrized curves

If x and y are given as functions of t, as happens with a parametrized curve in 2 dimensions, then the formulas for derivatives of y with respect to x, in terms of the derivatives of x and y with respect to t, ought to fall directly out of the notation. Unfortunately, the usual notation for higher derivatives prevents this.

To see how this should work, consider the first derivative. There, the formula is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}.$$

That is, simply divide both sides of the fraction by $\mathrm{d}t$. Another even slicker way to do this would be to reinterpret the differentials as derivatives with respect to t; that is, writing a dot above a quantity to indicate differentiation with respect to t, write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}}.$$

But If you try to do this with second derivatives, based on the usual notation for them, then you get a formula which is *wrong*:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \frac{\ddot{y}}{\dot{x}^2} = \frac{\mathrm{d}^2y/\mathrm{d}t^2}{(\mathrm{d}x/\mathrm{d}t)^2}.$$

(Here, I've written '$\neq$' to show that '$=$' would have been wrong, but it's possible that these may happen to be equal in certain examples.)

To get the correct formula instead, we simply need to differentiate $\dot{y}/\dot{x}$ using the Quotient Rule:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\dot{x}\,\mathrm{d}\dot{y}/\mathrm{d}t - \dot{y}\,\mathrm{d}\dot{x}/\mathrm{d}t}{\dot{x}^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}.$$

Dividing by $\dot{x}$ to change $\mathrm{d}/\mathrm{d}t$ to $\mathrm{d}/\mathrm{d}x$, the second derivative of y with respect to x is

$$(\mathrm{d}/\mathrm{d}x)^2 y = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} = \frac{\ddot{y}}{\dot{x}^2} - \frac{\dot{y}}{\dot{x}} \cdot \frac{\ddot{x}}{\dot{x}^2};$$

in other words, the naïve formula is only the first term of a two-term expression. This formula is a little long, but it will correctly give you the second derivative of y with respect to x using the first and second derivatives of x and y with respect to t.

There is a symbol for the second derivative using differentials that can serve as a mnemonic for this. To get it, we again differentiate $\mathrm{d}y/\mathrm{d}x$ using the Quotient Rule, only now using the Quotient Rule for differentials rather than the Quotient Rule for derivatives:

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}x\,\mathrm{d}(\mathrm{d}y) - \mathrm{d}y\,\mathrm{d}(\mathrm{d}x)}{(\mathrm{d}x)^2} = \frac{\mathrm{d}x\,\mathrm{d}^2y - \mathrm{d}y\,\mathrm{d}^2x}{\mathrm{d}x^2}.$$

Dividing by $\mathrm{d}x$ to turn d into $\mathrm{d}/\mathrm{d}x$, the second derivative of y with respect to x is

$$(\mathrm{d}/\mathrm{d}x)^2 y = \frac{\mathrm{d}x\,\mathrm{d}^2y - \mathrm{d}y\,\mathrm{d}^2x}{\mathrm{d}x^3} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2x}{\mathrm{d}x^2}.$$

As you can see, replacing 'd's with dots throughout gives the formula from the previous paragraph again.

For this reason, I don't like to write $\mathrm{d}^2y/\mathrm{d}x^2$ for the second derivative of y with respect to x. Of course, nobody wants to write the formula from the previous paragraph when they just want a symbol for the second derivative; fortunately, you can write $(\mathrm{d}/\mathrm{d}x)^2 y$ for that. This simply means that you apply the operation $\mathrm{d}/\mathrm{d}x$ (find the differential and then divide by $\mathrm{d}x$, or equivalently find the derivative with respect to x) twice to get the second derivative, which is certainly correct. You can even use this as a mnemonic for finding this second derivative: instead of interpeting $\mathrm{d}/\mathrm{d}x$ as taking the differential and then dividing by $\mathrm{d}x$, interpret it as taking the derivative with respect to t and then dividing by $\dot{x}$. This is essentially how the textbook tells you to take the second derivative.

Finally, whether you use either $(\mathrm{d}x\,\mathrm{d}^2y - \mathrm{d}y\,\mathrm{d}^2x)/\mathrm{d}x^3$ or $(\mathrm{d}/\mathrm{d}x)^2y$, either way you can perform practical caclulations by interpreting the differentials literally. You simply have to write everything in terms of t, put $\mathrm{d}t$ and d^2t in where they naturally appear, and find that the differentials of t cancel in the final answer. Alternatively, anticipating that the differentials of t will cancel, you can ignore them, which turns taking differentials into taking derivatives with respect to t again.

I'll do Example 10.2.2 on page 565 of the textbook to illustrate all of these approaches. (In that example, $x = t - t^2$, $y = t - t^3$, and you are asked to find $(\mathrm{d}/\mathrm{d}x)^2y$.) First, $\mathrm{d}x = \mathrm{d}t - 2t\,\mathrm{d}t$, or $\dot{x} = \mathrm{d}x/\mathrm{d}t = 1 - 2t$. Next, $\mathrm{d}^2x = \mathrm{d}^2t - 2\,\mathrm{d}t^2 - 2t\,\mathrm{d}^2t$ (whre I've applied the Product Rule to the second term of $\mathrm{d}x$), while $\ddot{x} = -2$. Similarly, $\mathrm{d}y = \mathrm{d}t - 3t^2\,\mathrm{d}t$, or $\dot{y} = 1 - 3t^2$. Next, $\mathrm{d}^2y = \mathrm{d}^2t - 6t\,\mathrm{d}t^2 - 3t^2\,\mathrm{d}^2t$, while $\ddot{y} = -6t$.

Now, to find $(\mathrm{d}/\mathrm{d}x)y = \mathrm{d}y/\mathrm{d}x$, either directly divide $(\mathrm{d}t - 3t^2\,\mathrm{d}t)/(\mathrm{d}t - 2t\,\mathrm{d}t)$ and simplify this (by cancelling factors of $\mathrm{d}t$) to $(1 - 3t^2)/(1 - 2t)$, or instead divide $\dot{y}/\dot{x}$, which again gives $(1 - 3t^2)/(1 - 2t)$. (This is pretty much the same process, no matter how you go about it.) Then to find $(\mathrm{d}/\mathrm{d}x)^2y$, one way is to differentiate $(\mathrm{d}/\mathrm{d}x)y$ (found in the previous paragraph) with respect to x again. Either take

$$\frac{\mathrm{d}\left(\frac{1-3t^2}{1-2t}\right)}{\mathrm{d}x} = \frac{\frac{2\,\mathrm{d}t - 6t\,\mathrm{d}t + 6t^2\,\mathrm{d}t}{(1-2t)^2}}{\mathrm{d}t - 2t\,\mathrm{d}t}$$

and simplify by cancelling factors of $\mathrm{d}t$, or take

$$\frac{(\mathrm{d}/\mathrm{d}t)\left(\frac{1-3t^2}{1-2t}\right)}{\dot{x}} = \frac{\frac{2-6t+6t^2}{(1-2t)^2}}{1-2t};$$

either way, you get

$$(\mathrm{d}/\mathrm{d}x)^2y = \frac{2 - 6t + 6t^2}{(1-2t)^3}.$$

This is essentially how the textbook does this problem.

Alternatively, using the formula

$$(\mathrm{d}/\mathrm{d}x)^2y = \frac{\mathrm{d}x\,\mathrm{d}^2y - \mathrm{d}y\,\mathrm{d}^2x}{\mathrm{d}x^3},$$

we immediately get

$$\frac{(\mathrm{d}t - 2t\,\mathrm{d}t)(\mathrm{d}^2t - 6t\,\mathrm{d}t^2 - 3t^2\,\mathrm{d}^2t) - (\mathrm{d}t - 3t^2\,\mathrm{d}t)(\mathrm{d}^2t - 2\,\mathrm{d}t^2 - 2t\,\mathrm{d}^2t)}{(\mathrm{d}t - 2t\,\mathrm{d}t)^3},$$

which simplifies drastically to the same answer as above. (Notice that there is no need to work out $\mathrm{d}y/\mathrm{d}x$ first!) Or using

$$(\mathrm{d}/\mathrm{d}x)^2y = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3},$$

you immediately get

$$\frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^3},$$

which simplifies (somewhat less drastically) to the same answer once again. I prefer this last method, which gets the answer in one step after the preliminary calculations and doesn't require quite as much algebra to simplify as the corresponding method using differentials.

Arclength

When finding the length of a curve by integration, you are ultimately integrating an expression such as $\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$. This particular expression applies in 2 dimensions; in words, it is the principal square root of the sum of the square of the differential of x and the square of the differential of y. An expression like this, containing differentials, is called a *differential form*; the textbook mentions differential forms briefly in Section 15.3, but they are really all over the place in multivariable Calculus, sometimes hidden just under the surface, sometimes out in the open without being acknowledged.

This particular differential form is called the **arclength element** and is traditionally written $\mathrm{d}s$ (although that notation is misleading for reasons that I will return to next term). A simpler way to think of $\mathrm{d}s$, which works in *any* number of dimensions, is as $\|\mathrm{d}P\|$, the magnitude of the differential of the position P. Remember that $\mathrm{d}P$ is a vector when P is a point, so it has a magnitude; in fact, $\mathrm{d}P$ is the same as $\mathrm{d}\mathbf{r}$ (where $\mathbf{r} = P - \mathrm{O}$), so you can also think of $\mathrm{d}s$ as $\|\mathrm{d}\mathbf{r}\|$, the magnitude of the differential of the position vector $\mathbf{r}$. In 2 dimensions, where $P = (x, y)$ and $\mathbf{r} = \langle x, y\rangle$, $\mathrm{d}\mathbf{r} = \mathrm{d}P = \langle \mathrm{d}x, \mathrm{d}y\rangle$, whose magnitude is the arclength element that I talked about above. In 3 dimensions, $\mathrm{d}P = \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\rangle$, whose magnitude is $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$.

When working with a parametrized curve, every variable (x and y, and z if it exists, whether individually or combined into P or $\mathbf{r}$) is given as a function of some parameter t. By differentiating these, their differentials come to be expressed using t and $\mathrm{d}t$. The absolute value $|\mathrm{d}t|$ will naturally appear in the integrand; but if you set up the integral so that t is increasing, then $\mathrm{d}t$ is positive, so $|\mathrm{d}t| = \mathrm{d}t$. Then you can write $\|\mathrm{d}P\|$ as $\|\mathbf{v}\|\,|\mathrm{d}t| = \|\mathbf{v}\|\,\mathrm{d}t$, where $\mathbf{v} = \mathrm{d}P/\mathrm{d}t = \mathrm{d}\mathbf{r}/\mathrm{d}t$ is the velocity, as given in the textbook. More explicitly, this is

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

(in 2 dimensions), which is also given in the textbook. But while you might integrate this in practice to perform a specific calculation, you are most fundamentally integrating a differential form in which t does not appear. This is why the result ultimately does not depend on how you parametrize the curve. (Next term, I'll discuss what it means, in general, to integrate a differential form along a curve, including why and to what extent this is independent of any parametrization.)