Differential equations

A **differential equation** is an equation with differentials or derivatives in it. Here are three examples of differential equations:

$$f'(x) = 3f(x);$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3y;$$
$$\mathrm{d}y = 3y\,\mathrm{d}x.$$

In fact, these three examples are all basically equivalent. If you are given the first of these, then you should make up a name for $f(x)$, say y, and turn the first equation into the middle one. And in the middle equation, you should clear fractions to turn it into the last one. (But any of these might be the original form, depending on how the equation is thought up in the first place.)

To actually solve this equation, you can use the technique of **separation of variables**. After reaching the last equation, notice that x only appears on the right-hand side but y appears on both sides. If you divide both sides by y, however, then y appears only on the left-hand side. (If $y = 0$, then dividing by y is invalid; I'll come back to that later.) Then the variables are separated:

$$\frac{\mathrm{d}y}{y} = 3\,\mathrm{d}x.$$

(If you're ever unsure which side to put which variable on, then try to put the differentials in the numerators of any fractions. In this example, $1/\mathrm{d}x = 3y/\mathrm{d}y$ would have the variables separated just as much, but it would be less useful, because the next step, below, wouldn't work.)

Now take the indefinite integral of each side of the equation:

$$\int \frac{\mathrm{d}y}{y} = \int 3\,\mathrm{d}x;$$
$$\ln|y| + C_1 = 3x + C_2;$$
$$\ln|y| = 3x + C_2 - C_1.$$

Each integral gives an arbitrary constant, and I subtracted to put them both on the right-hand side. However, since $C_2 - C_1$ could itself be any constant, you can just write this as

$$\ln|y| = 3x + C.$$

In practice, you can skip the other steps with constants and just remember to tack a constant onto the last integral in the equation.

We're not finished; this equation is no longer a differential equation, but it also hasn't been solved for anything. If we want to solve it for y, then we still need to do some algebra to get y by itself on its side of the equation:

$$|y| = \mathrm{e}^{3x+C};$$
$$y = \pm\mathrm{e}^{3x+C}.$$

(If you're given an equation in x and y, then it's a good bet that they want you to solve for y; if you're given an equation like the first example with a function in it, then it's a good bet that they want you to solve for the function. But in principle, you could solve any of these equations for x instead.)

There is one mistake here, which is the step where I divided by y. If $y = 0$, then this is invalid. Furthermore, if $y = 0$ always, then the equation is true, because then both sides of the original equation (in any of the three forms) are 0. (This sort of special exception is fairly common with differential equations.) So a complete solution is

$$y = \pm\mathrm{e}^{3x+C} \text{ or } y = 0.$$

You can make the final solution look a bit nicer by writing $\pm e^{3x+C}$ as $\pm e^C e^{3x}$ and then making up a name for $\pm e^C$, say P. Since e^C could be any positive number, P could be any positive or negative number; the exception $y = 0$ is captured by $P = 0$. So the nicest form of the final solution is

$$y = Pe^{3x},$$

where P is an arbitrary constant. (However, you shouldn't always expect to be able to do a simplifying trick like that.)

Of course, if the original form of the equation is the first example, then you should write this solution as

$$f(x) = Pe^{3x}.$$

Initial-value problems

An **initial-value problem** consists of a differential equation together with enough data to determine the arbitrary constants. Here are three examples of initial-value problems:

$$\begin{aligned} f'(x) &= 3f(x),\ f(0) = 5; \\ \frac{dy}{dx} &= 3y,\ y|_{x=0} = 5; \\ dy &= 3y\,dx,\ y|_{x=0} = 5. \end{aligned}$$

Again, these three examples are all basically equivalent; if $y = f(x)$, then $y|_{x=0}$ means $f(0)$.

There are two ways to solve an initial-value problem. One is to ignore the initial value and just solve the differential equation, at first. In this example, that gives us

$$y = Pe^{3x},$$

as you've seen. Then you put in the given values, which in this case gives

$$5 = Pe^{3(0)}.$$

Now you can solve for P:

$$\begin{aligned} 5 &= P(1); \\ P &= 5. \end{aligned}$$

Therefore, the final answer to the initial-value problem is

$$y = 5e^{3x}.$$

(Again, if the original form of the equation is the first example, then you should write this solution as $f(x) = 5e^{3x}$.)

Another technique is to solve the entire problem at once with the help of semidefinite integrals. Besides the *definite* integral $\int_a^b f(x)\,dx$ and the *indefinite* integral $\int f(x)\,dx$, there is also a **semidefinite integral** $\int_a f(x)\,dx$. While the definite integral works out to a specific value (as long as f, a, and b are specified), the indefinite and semidefinite integrals still have the variable x in them. On the other hand, while the indefinite integral depends on an arbitrary C, the definite and semidefinite integrals don't have this. So the semidefinite integral fits in between the other two kinds.

Here is one way to define it:

$$\int_{x=a} f(x)\,dx = \int_{t=a}^{x} f(t)\,dt.$$

That is, introduce a new variable t and use the old variable x as the upper bound of a definite integal. The Second Fundamental Theorem of Calculus,

$$\int_{x=a}^{b} f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_{x=a}^{b} = \left(\int f(x)\,dx\right)\Big|_{x=b} - \left(\int f(x)\,dx\right)\Big|_{x=a},$$

also tells us how to evaluate semidefinite integrals:

$$\int_{x=a} f(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x - \Big(\int f(x)\,\mathrm{d}x\Big)\Big|_{x=a}.$$

In other words, work out the indefinite integral as usual; then, instead of evaluating this at two values of the variable before subtracting, evalute it at one value and keep the variable in the other expression (then subtract). For example,

$$\int_{x=1} x\,\mathrm{d}x = \frac{x^2}{2} - \left(\frac{x^2}{2}\right)\bigg|_{x=1} = \frac{x^2}{2} - \left(\frac{(1)^2}{2}\right) = \frac{1}{2}x^2 - \frac{1}{2}.$$

(You can probably skip the step with $|_{x=1}$ in it, since once you've written down $x^2/2$ before the minus sign, you can immediately plug in 1 for x to get $(1)^2\big/2$ after the minus sign.)

Let's solve the example

$$\mathrm{d}y = 3y\,\mathrm{d}x,\ y|_{x=0} = 5$$

using semidefinite integrals. Again, separate the variables:

$$\frac{\mathrm{d}y}{y} = 3\,\mathrm{d}x.$$

Now instead of taking *indefinite* integrals of both sides, take *semidefinite* integrals, using the initial value to guarantee that you're doing the same thing to each side even though it's being done using different variables. In this case, since $y = 5$ when $x = 0$, a semidefinite integral starting at $y = 5$ is the same operation as a semidefinite integral starting at $x = 0$, so

$$\int_{y=5} \frac{\mathrm{d}y}{y} = \int_{x=0} 3\,\mathrm{d}x.$$

Evaluating these using the FTC gives

$$\ln|y| - \ln|5| = 3x - 3(0).$$

So compared to the integration without the initial value, the difference is that we know which specific constants to use in each integral. Now again, solve for y to finish:

$$\begin{aligned} \ln|y| &= 3x - 0 + \ln 5; \\ |y| &= \mathrm{e}^{3x+\ln 5}; \\ y &= \pm 5\mathrm{e}^{3x}. \end{aligned}$$

This is not completely perfect, because of the $\pm$, but we can figure this out by checking whether y really is 5 when $x = 0$; this will only be true if the sign is $+$. Finally, since we did again divide by y while solving this, check to make sure that y is never zero in the solution; it's not, so the final answer is

$$y = 5\mathrm{e}^{3x}.$$

Of course, this is the same solution as we got before, but this time we got the entire solution all at once without having to first get a solution with an arbitrary constant and then solving for the constant. You may solve intial-value problems using whichever method you prefer.

Integrals as solutions to equations

Although we normally solve a differential equation by taking integrals, you can also think of an integral as a solution to a differential equation. For example, the indefinite integral $\int f(x)\,\mathrm{d}x$ is the solution to the differential equation $\mathrm{d}y/\mathrm{d}x = f(x)$, and the semidefinite integral $\int_{x=a} f(x)\,\mathrm{d}x$ is the solution to the initial-value problem $(\mathrm{d}y/\mathrm{d}x = f(x), y|_{x=a} = 0)$. More generally, the solution to the initial-value problem $(\mathrm{d}y/\mathrm{d}x = f(x), y|_{x=a} = c)$ is $\int_{x=a} f(x)\,\mathrm{d}x + c$. These kinds of initial-value problems are in Sections 4.8 and 5.5 of the textbook and are covered in Calculus 1; more general differential equations and initial-value problems are in Section 7.2 and are covered in Calculus 2.

(There are even more general differential equations than I have discussed here, ones in which it is impossible the separate the variables in the equation; some of these are covered in Chapters 16 and 17 of the online-only version of the textbook. Yet more general differential equations are covered in SCC's course on differential equations, which is basically Calculus 4, but using a different textbook dedicated to that subject. Beyond that, there are graduate-level courses that you could take at a university; in fact, the study of differential equations is a major field of active research in mathematics. We are very far from knowing how to solve them all!)