Integration by parts

Integration by parts is based on the Product Rule for differentiation. In terms of differentials, the Product Rule says that $\mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v$. Taking indefinite integrals of both sides and rearranging the terms slightly, this becomes

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Unlike integration by substitution, you don't rewrite the problem in terms of u (nor v). Instead, you identify suitable u and v and their differentials and then write out the equation above in terms of x (or whatever your variable is).

You want to pick u and v so that $\int u\,\mathrm{d}v$ is the integral that you care about, which means splitting up the factors of the integrand, some into u and some into $\mathrm{d}v$. Once you know u and $\mathrm{d}v$, you can find $\mathrm{d}u$ and v, at least if you know how to integrate whatever $\mathrm{d}v$ is. (When you do this integration of $\mathrm{d}v$ to get v, you have a choice up to a local constant; you're deciding what v is, so just pick the simplest expression.) If you split things up well, then $\int v\,\mathrm{d}u$ will be simpler than what you started with.

Here is my advice on how to split factors into u and $\mathrm{d}v$ so that integration by parts will make the next integral easier. The items on the top of the list are the best choices for $\mathrm{d}v$, and the items on the bottom are the best choices for u. Put as many factors as you can into $\mathrm{d}v$, starting at the top of this list and working your way to the bottom, as long as you still have something that you know how to integrate to get v. Then put whatever factors are left over into u.

- $\mathrm{d}x$ (this *must* go into $\mathrm{d}v$),
- $\mathrm{e}^x$and other exponential expressions,
- $\sin x$ and other trigonometric expressions,
- polynomials and other algebraic expressions,
- $\ln x$ and other logarithmic expressions,
- $\operatorname{asin} x = \sin^{-1} x$ and other inverse trigonometric expressions.

In complicated cases, you may have to use integration by parts more than once. Just keep going until either you get something that you can handle or you get back to where you started. In the latter case, you can set up an equation to solve for your integral.