One of the basic applications of vector calculus —arguably the original application— is the classical theory of electromagnetic fields that was fully worked almost 150 years ago by James Clerk Maxwell. Maxwell's equations of electromagnetism have been expressed in many formalisms over the years: explicitly using partial derivatives of component functions (the way Maxwell presented them), using quaternions (like complex numbers with three imaginary dimensions, which is how Maxwell really thought of them), using the vector calculus of Oliver Heaviside and Willard Gibbs (the simplification of quaternionic calculus that is taught in the course textbook), using differential forms in three-dimensional space (which is how I usually think of them), and using differential forms in four-dimensional space-time. Each is simpler and more elegant than the last.

All of the differential forms appearing in these notes will be oriented or pseudo-oriented exterior differential forms. To keep the notation simple, *I will leave out the symbol '$\wedge$' in the wedge product and the exterior derivative.* So if you see two differentials (or differential forms) multiplied together, then they're being multiplied by the wedge product; and if you see the differential of a differential form, then it's the exterior differential. Also, unlike in the specific problems that you've done in this course, I'll use variables that refer directly to differential forms; typically, these variables with be in a fancy calligraphic font.

The quantities in the equations

To be very definite, I will give operational definitions of the physical quantities that appear in Maxwell's equations, describing how you would (in principle) measure them.

I will take as a basic notion the idea of **electric charge**. Electric charge may be positive or negative, and the difference between these is perfectly arbitrary (which is in some ways similar to the right-hand rule); what's important is that there is a difference, and positive and negative charges cancel each other out. In any given region of space, there is a certain total charge in that region, which we'll assume is given by integrating a continuous rank-3 pseudo-oriented differential form, the **charge form** $\mathcal{Q}$. (The existence of this differential form is actually a theorem, under certain assumptions about additivity and continuity of charge.) We may write

$$\mathcal{Q} = \rho\, \text{đ}V = \rho\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

where the scalar field ρ is the **charge density**. The SI unit of charge is the coulomb (named after Charles-Augustin de Coulomb, who discovered the inverse-square law of static electricity); charge density is measured in coulombs per cubic metre.

Together with electric charge, we have **electric current**, which is the flow of electric charge. We measure current through a pseudo-oriented surface; the total rate (with respect to time) at which charge moves through the surface in the given direction is the current through that surface. (Negative charge moving through the surface in the negative direction counts positively, like positive charge moving in the positive direction; negative charge moving in the positive direction and positive charge moving in the negative direction count negatively.) The current through a pseudo-oriented surface is given by integrating a continuous rank-2 pseudo-oriented form, the **current form** $\mathcal{J}$. We may write

$$\mathcal{J} = \mathbf{J} \cdot \text{đ}\mathbf{S} = J_1\, \mathrm{d}y\, \mathrm{d}z + J_2\, \mathrm{d}z\, \mathrm{d}x + J_3\, \mathrm{d}x\, \mathrm{d}y,$$

where the vector field $\mathbf{J}$ is the **current density**. The SI unit of current is the coulomb per second, or ampere (named after André-Marie Ampère, who discovered Ampere's Law, discussed below); current density is measured in amperes per square metre.

Based on these, we can now define some other quantities. When the work (transfer of energy) done on a charged object is proportional to its charge, we consider that the work is done by an *electric field.* If a charged object travels through an electric field along an oriented curve, then the work done on the particle is the product of the particle's charge and the **electric potential** along the curve. Since the charge on any actual object is spread out over space and charge (as you'll see below) also affects the electric field, we really need to consider the limiting case of an object with both infinitesimal volume and infinitesimal charge density. The electric potential along an oriented curve is given by integrating a continuous rank-1 oriented form, the **electric potential form** $\mathcal{E}$. We may write

$$\mathcal{E} = \mathbf{E} \cdot \mathrm{d}\mathbf{r} = E_1\,\mathrm{d}x + E_2\,\mathrm{d}y + E_3\,\mathrm{d}z,$$

where the vector field $\mathbf{E}$ is the **electric field strength**. The SI unit of electric potential is the joule per coulomb, or volt; electric field strength is measured in volts per metre.

The electric field not only affects charges but also is created by them. As charges move in response to the work done on them by the electric field, this tends to cancel out the original field. (This is a general theme in electromagnetism, that any phenomenon has effects that counteract the original cause.) In particular, if a sheet of material that conducts electric current (a *Faraday shield*) is placed in an electric field, then the free charged particles in the shield will move to opposite sides, blocking out the electric field in the interior of the sheet. The **electric flux** through a pseudo-oriented surface is the total charge induced by the electric field on the outside of a continuous Faraday shield along that surface (or opposite the charge induced on the inside of the shield). Again, we must really consider a limiting case, that of a sheet with infinitesimal thickness and infinite conductance. The electric flux through a pseudo-oriented surface is given by integrating a continuous rank-2 pseudo-oriented form, the **electric flux form** $\mathcal{D}$. We may write

$$\mathcal{D} = \mathbf{D} \cdot \text{đ}\mathbf{S} = D_1\,\mathrm{d}y\,\mathrm{d}z + D_2\,\mathrm{d}z\,\mathrm{d}x + D_3\,\mathrm{d}x\,\mathrm{d}y,$$

where the vector field $\mathbf{D}$ is the **electric displacement**. The SI unit of electric flux is the coulomb again; electric displacement is measured in coulombs per square metre.

Besides the electric field, there is also a *magnetic field*. Although this may be thought of as dealing with magnetic poles (instead of electric charges), magnetic poles are not individual objects but always come in pairs. We now understand (and Maxwell already understood) that magnetism deals with electric currents, with a north pole and a south pole appearing on either side of a rotating current. If a wire with current flowing through it travels through a magnetic field, then it traces out a surface, which we orient (not pseudo-orient!) as the direction of travel followed by the direction of the current. Then the work done on the wire is the product of the wire's current and the **magnetic flux** on the surface. Since any actual conducting wire has some thickness and current (as you'll see below) also affects the magnetic field, we really need to consider the limiting case of a wire with both infinitesimal thickness and infinitesimal current density. The magnetic flux on an oriented surface is given by integrating a continuous rank-2 oriented form, the **magnetic flux form** $\mathcal{B}$. We may write

$$\mathcal{B} = \mathbf{B} \cdot \text{đ}\mathbf{S} = B_1\,\mathrm{d}y\,\mathrm{d}z + B_2\,\mathrm{d}z\,\mathrm{d}x + B_3\,\mathrm{d}x\,\mathrm{d}y,$$

where the pseudo-vector field $\mathbf{B}$ is the **magnetic flux density**. The SI unit of magnetic flux is the joule per ampere, or weber; magnetic flux density is measured in webers per square metre, or teslas.

Just as the electric field causes charges to move to counteract it, so the magnetic field creates currents that counteract it. In particular, if a tube of conductive material (a *solenoid*) is placed in a magnetic field, then the field will induce a current on the inside of the solenoid, blocking the magnetic field within the solenoid. The **magnetic potential** around a pseudo-oriented curve (not oriented!) is the total current induced by the magnetic field in a continuous solenoid surrounding the curve in the direction opposite the curve's pseudo-orientation. Once more, we must really consider a limiting case, that of a tube with infinitesimal radius and infinite conductance. The magnetic potential around a pseudo-oriented curve is given by integrating a continuous rank-1 pseudo-oriented form, the **magnetic potential form** $\mathcal{H}$. We may write

$$\mathcal{H} = \mathbf{H} \cdot \mathrm{d}\mathbf{r} = H_1\,\mathrm{d}x + H_2\,\mathrm{d}y + H_3\,\mathrm{d}z,$$

where the pseudo-vector field $\mathbf{H}$ is the **magnetizing field strength**. The SI unit of magnetic potential is the ampere again; magnetizing field strength is measured in amperes per metre.

The constitutive relations

Before I get to the four equations generally called Maxwell's, I need to clear something up. We have two ways to measure an electric field, the electric potential along a curve (the integral of $\mathcal{E}$) and the electric flux through a surface (the integral of $\mathcal{D}$); similarly, we have two ways to measure a magnetic field, the magnetic flux on a surface (the integral of $\mathcal{B}$) and the magnetic potential around a curve (the integral of $\mathcal{H}$). Since $\mathcal{E}$ and $\mathcal{D}$ measure the same physical field, there should be a relationship between them, and the same for $\mathcal{B}$ and $\mathcal{H}$. The simplest relationship would be that each of these quantities is the Hodge dual of its partner; after all, the Hodge dual of an oriented 1-form is a pseudo-oriented 2-form, etc. (Then we would also have $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$.) However, there are a few complications with that.

First, if we measure $\mathcal{D}$ and $\mathcal{H}$ with actual conducting materials, then (even in the limit of infinite conductance!) there will always be charges that are bound in the material, unable to be moved by the fields, and there will also be bound currents sometimes (as in a magnet). Thus, $\mathcal{D}$ and $\mathcal{H}$ effectively measure only the free charge and current. When people express Maxwell's equations using only $\mathcal{E}$ and $\mathcal{B}$ instead, they speak of Maxwell's equations *in a vacuum*.

Secondly, even in vacuum, $\mathcal{E}$ and $\mathcal{D}$ are measured in different units (and similarly for $\mathcal{B}$ and $\mathcal{H}$). Up to a point, this is expected; since volume has units of cubic metres, we expect the Hodge dual to affect units. However, this only affects units of length, and we need more than that (in particular, the units of charge are reversed). In vaccum, the unit conversion is done by fundamental physical constants, the *electric constant* ϵ_0 and the *magnetic constant* μ_0; then we have

$$*\mathcal{E} = \frac{\mathcal{D}}{\epsilon_0}$$

(so $*\mathcal{D} = \epsilon_0\mathcal{E}$) and

$$*\mathcal{B} = \mu_0\mathcal{H}$$

(so $*\mathcal{H} = \mathcal{B}/\mu_0$). Ultimately, the SI units are defined so that ϵ_0 is exactly

$$\frac{2^3 5^7}{7^2 73^2 293339^2 \pi} \approx 8.85 \times 10^{-12}$$

farads per metre and μ_0 is exactly

$$\frac{\pi}{2^5 5^7} \approx 1.26 \times 10^{-6}$$

henries per metre. (A farad is a square coulomb per joule, named after Michael Faraday, who discovered Faraday's Law, below; a henry is a joule per square ampere. By the way, there are only two more SI units related specifically to electromagnetism: the siemens is a farad per second, and the ohm is a henry per second. But we will not need these here.)

In a medium, we typically have $*\mathcal{E} = \mathcal{D}/\epsilon$ and $*\mathcal{B} = \mu\mathcal{H}$ (or $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ in terms of vector fields) for some constants ϵ and μ, the *permittivity* and *permeability* of the medium. (Then ϵ_0 and μ_0 are respectively the permittivity and permeability of the vacuum.) Sometimes things are not so simple (for example, the permittivity or permeability may depend on the direction); but we always have some relationship between these quantities, called the *constitutive relations* of the material. When we use differential forms instead of vector fields, the constitutive relations are the *only* equations in which the Hodge dual operator appears, hence the only place where geometric ideas (such as length, angle, and volume) play a role; using vector fields obscures this fact.

Static systems

Maxwell found four equations, which I will state first for *static* systems, that is those in which the distribution of charges, currents, and fields does not change with time. In a static system, the total current through the boundary of any region of space must be zero, because otherwise the total charge inside that region would be changing; this is the *continuity equation*

$$\int_{\partial Q} \mathcal{J} = 0,$$

which is not counted as one of Maxwell's four. Assuming that $\mathcal{J}$ is continuously differentiable, then the Stokes Theorem turns this into $\int_Q \mathrm{d}\mathcal{J} = 0$; since this holds for any region Q, we conclude that

$$\mathrm{d}\mathcal{J} = 0,$$

which is $\nabla \cdot \mathbf{J} = 0$ in terms of the current density. Like the continuity equation, each of Maxwell's equations will have an integral and differential form.

The simplest of Maxwell's equations is

$$\int_{\partial Q} \mathcal{B} = 0,$$

stating that the magnetic flux through the boundary of any region in space is zero. In other words, magnetic flux, like current in a static system, flows continuously with no sink or source. The differential form is

$$\mathrm{d}\mathcal{B} = 0,$$

or $\nabla \cdot \mathbf{B} = 0$ in vector calculus.

Similarly, *Faraday's Law* for static systems states that the electric potential along the boundary of any oriented surface is zero:

$$\int_{\partial R} \mathcal{E} = 0.$$

In differential form, this becomes

$$\mathrm{d}\mathcal{E} = 0,$$

which is $\nabla \times \mathbf{E} = 0$ in vector calulus. Thus, $\mathcal{E}$ is an exact differential, and $\mathbf{E}$ is a conservative vector field.

Next, Gauss's Law (after Carl Gauß) states that the total electric flux outward through the boundary of any region in space equals the total electric charge contained in that region:

$$\int_{\partial Q} \mathcal{D} = \int_Q \mathcal{Q}.$$

In differential form,

$$\mathrm{d}\mathcal{D} = \mathcal{Q};$$

in vector calculus, $\nabla \cdot \mathbf{D} = \rho$. Thus, unlike magnetic flux, electric flux has sources and sinks, which are electric charges.

Finally, Ampere's Law for static systems states that the magnetic potential around the boundary of a pseudo-oriented surface equals the total current through that surface:

$$\int_{\partial R} \mathcal{H} = \int_R \mathcal{J}.$$

In differential form,

$$\mathrm{d}\mathcal{H} = \mathcal{J};$$

in vector calculus, $\nabla \times \mathbf{H} = \mathbf{J}$. Thus, currents are sources for the magnetic field.

The reason that the continuity equation is not counted as one of Maxwell's equations is that it actually follows from Ampere's Law. Specifically (in a static system), we have

$$\int_{\partial Q} \mathcal{J} = \int_{\partial\partial Q} \mathcal{H} = 0,$$

since the boundary of a boundary is empty.

Electrodynamics

Some of the equations above only apply when the charges, currents, and fields don't change with time. Maxwell's equations also come in a more general form that drops this assumption. It is easy enough to state the integral forms of these equations, but the differential forms require taking seriously the four-dimensional nature of our universe in space and time. In vector calculus, this is done by treating space and time separately, but differential forms make sense in any number of dimensions; this ultimately simplifies Maxwell's equations. Finally, the constitutive relations in 4 dimensions clarify the nature of the geometry of spacetime in our universe, which leads naturally to Albert Einstein's special theory of relativity.

Here are Maxwell's equations in integral form:

$$\begin{aligned}
&\int_{\partial Q} \mathcal{B} = 0, \\
&\int_{\partial R} \mathcal{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_R \mathcal{B}, \\
&\int_{\partial Q} \mathcal{D} = \int_Q \mathcal{Q}, \\
&\int_{\partial R} \mathcal{H} = \int_R \mathcal{J} + \frac{\mathrm{d}}{\mathrm{d}t}\int_R \mathcal{D}.
\end{aligned}$$

In words, the magnetic flux on the boundary of an oriented region of space is still zero, but the electric potential along the boundary of an oriented surface is now the opposite of the rate of change with time of the magnetic flux on that surface. Similarly, the electric flux out of the boundary of a region of space is still the total electric charge in that region, but the magnetic potential around the boundary of a pseudo-oriented surface is now the sum of the electric current through that surface and the rate of change with time of the electric flux through that surface. The continuity equation (which now relies on both Ampere's Law and Gauss's Law) becomes

$$\int_{\partial Q} \mathcal{J} = \int_{\partial\partial Q} \mathcal{H} - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial Q} \mathcal{D} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_Q \mathcal{Q};$$

in words, if current flows out of the boundary of a region of space, then the total charge in that region goes down accordingly. (The reason that we credit these equations to Maxwell, when all of them are laws discovered earlier by other people, is that Ampère didn't know about the contribution of $\mathcal{D}$ to his law; Maxwell realized that it had to be there to get the correct continuity equation, and this is what made the system complete.)

If we separate space from time, writing d_s for the exterior differential on space (so holding time t constant) and using a dot to indicate differentiation with respect to time, then here are the equations in differential form:

$$\begin{aligned}
\mathrm{d}_s\mathcal{B} &= 0, \\
\mathrm{d}_s\mathcal{E} &= -\dot{\mathcal{B}}, \\
\mathrm{d}_s\mathcal{D} &= \mathcal{Q}, \\
\mathrm{d}_s\mathcal{H} &= \mathcal{J} + \dot{\mathcal{D}}.
\end{aligned}$$

The continuity equation in differential form is

$$\mathrm{d}_s\mathcal{J} = -\dot{\mathcal{Q}}.$$

Rewriting in vector calculus (which is how you usually find Maxwell's equations on T-shirts):

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \cdot \mathbf{D} &= \rho, \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t};
\end{aligned}$$

the continuity equation is

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

This is a little unsatisfying, because differential forms are supposed to take care of *all* variation of a quantity, which in this context is variation in both space and time. In general, we have $\mathrm{d}\omega = \mathrm{d}_s\omega + \dot{\omega}\,\mathrm{d}t$, for ω any differential form defined on spacetime. Then $\mathrm{d}(\omega\,\mathrm{d}t) = \mathrm{d}\omega\,\mathrm{d}t = (\mathrm{d}_s\omega + \dot{\omega}\,\mathrm{d}t) = \mathrm{d}_s\omega\,\mathrm{d}t + 0 = \mathrm{d}_s\omega\,\mathrm{d}t$ (since $\mathrm{d}t\,\mathrm{d}t = 0$ with the wedge product). This works for $\mathcal{E}$, $\mathcal{H}$, and $\mathcal{J}$, since $\dot{\mathcal{E}}$, $\dot{\mathcal{H}}$, and $\dot{\mathcal{J}}$ never appear. In fact, it works out very nicely to multiply Faraday's Law and Ampere's Law by $\mathrm{d}t$. If we then add or subtract these equations from the ones that precede them, then we can make $\mathrm{d}\mathcal{B}$ and $\mathrm{d}\mathcal{D}$ appear as well. That is, the first pair adds as follows:

$$\begin{aligned} \mathrm{d}_s\mathcal{B} + \mathrm{d}_s\mathcal{E}\,\mathrm{d}t &= 0 - \dot{\mathcal{B}}\,\mathrm{d}t, \\ \mathrm{d}_s\mathcal{B} + \dot{\mathcal{B}}\,\mathrm{d}t + \mathrm{d}_s\mathcal{E}\,\mathrm{d}t &= 0, \\ \mathrm{d}\mathcal{B} + \mathrm{d}(\mathcal{E}\,\mathrm{d}t) &= 0, \\ \mathrm{d}F &= 0. \end{aligned}$$

In the last step, I've introduced

$$F = \mathcal{B} + \mathcal{E}\,\mathrm{d}t,$$

sometimes called the **Faraday form** (although the letter originally simply stood for 'field'). Similarly, the second pair subtracts as follows:

$$\begin{aligned} \mathrm{d}_s\mathcal{D} - \mathrm{d}_s\mathcal{H}\,\mathrm{d}t &= \mathcal{Q} - \mathcal{J}\,\mathrm{d}t - \dot{\mathcal{D}}\,\mathrm{d}t, \\ \mathrm{d}_s\mathcal{D} + \dot{\mathcal{D}}\,\mathrm{d}t - \mathrm{d}_s\mathcal{H}\,\mathrm{d}t &= \mathcal{Q} - \mathcal{J}\,\mathrm{d}t, \\ \mathrm{d}\mathcal{D} - \mathrm{d}(\mathcal{H}\,\mathrm{d}t) &= \mathcal{Q} - \mathcal{J}\,\mathrm{d}t, \\ \mathrm{d}M &= j. \end{aligned}$$

Now in the last step, I've introduced both the **Maxwell form**

$$M = \mathcal{D} - \mathcal{H}\,\mathrm{d}t$$

and the **four-current form**

$$j = \mathcal{Q} - \mathcal{J}\,\mathrm{d}t.$$

Let's take stock of where we are. We have an oriented rank-2 differential form F, measured in webers (which are the same as volt-seconds), a pseudo-oriented rank-2 differential form M, measured in coulombs (which are the same as ampere-seconds), and a pseudo-oriented rank-3 differential form j, also measured in coulombs. There are now only two Maxwell's equations:

$$\begin{aligned} \mathrm{d}F &= 0, \\ \mathrm{d}M &= j; \end{aligned}$$

the continuity equation is simply

$$\mathrm{d}j = 0.$$

We can also write these equations in integral form:

$$\begin{aligned} \int_{\partial R} F &= 0, \\ \int_{\partial R} M &= j; \\ \int_{\partial Q} j &= 0. \end{aligned}$$

Here, R is a 2-dimensional surface embedded in four-dimensional spacetime, which could be a surface as we normally think of it, for an instant, but is typically what we would think of as a curve, persisting through time (and perhaps moving, growing, or shrinking). Similarly, Q is a 3-dimensional hypersurface in spacetime, which could be a region of space for an instant but is typically what we would think of as a surface, again persisting and possibly changing through time. There is no vector-calculus form of these spacetime equations; neither F nor M can be described by vectors, even ones with 4 components (although there is a concept of bivector that could be used here if you really insist).

It's worth looking specifically at the components that would go into F, M, and j. We have

$$F = \mathcal{B} + \mathcal{E}\,\mathrm{d}t = \mathbf{B} \cdot \text{đ}\mathbf{S} + \mathbf{E} \cdot \mathrm{d}\mathbf{r}\,\mathrm{d}t = B_1\,\mathrm{d}y\,\mathrm{d}z + B_2\,\mathrm{d}z\,\mathrm{d}x + B_3\,\mathrm{d}x\,\mathrm{d}y + E_1\,\mathrm{d}x\,\mathrm{d}t + E_2\,\mathrm{d}y\,\mathrm{d}t + E_3\,\mathrm{d}z\,\mathrm{d}t;$$

this has 6 coefficients, containing all of the information in both $\mathcal{E}$ and $\mathcal{B}$ (so nothing is lost by combining the two equations into one). Similarly,

$$M = \mathcal{D} - \mathcal{H}\,\mathrm{d}t = \mathbf{D} \cdot \text{đ}\mathbf{S} - \mathbf{H} \cdot \mathrm{d}\mathbf{r}\,\mathrm{d}t = D_1\,\mathrm{d}y\,\mathrm{d}z + D_2\,\mathrm{d}z\,\mathrm{d}x + D_3\,\mathrm{d}x\,\mathrm{d}y - H_1\,\mathrm{d}x\,\mathrm{d}t - H_2\,\mathrm{d}y\,\mathrm{d}t - H_3\,\mathrm{d}z\,\mathrm{d}t,$$

and

$$j = \mathcal{Q} - \mathcal{J}\,\mathrm{d}t = \rho\,\text{đ}V - \mathbf{J} \cdot \text{đ}\mathbf{S}\,\mathrm{d}t = \rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - J_1\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t - J_2\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t - J_3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

(The information in the four-current form can be put into a four-dimensional vector, but I won't bother, since everything works already with forms.)