All of the integrals in vector calculus can be thought of as integrals of *differential forms* of one sort or another. Since integration of differential forms generalizes in ways that integration of vector fields cannot (some of which are important in applications, especially to physics), it's useful to be able to think about differential forms. Furthermore, one then needs fewer formulas for the various derivatives of vector fields and for the theorems that relate derivatives to integrals.

General principles

Here I spell out the general principles of integrating differential forms, but it's really the examples that follow that will make the ideas clear.

There are three sorts of differential forms that we'll need: *oriented* forms, *pseudo-oriented* forms, and *unoriented* forms. The oriented forms are the most straightforward kind and the simplest to calculate with. The pseudo-oriented forms are essentially the same as oriented forms, except that their sign is determined by using the right-hand rule; if we used the left-hand rule instead, then the pseudo-oriented forms would have opposite sign but the results of all integrals would stay the same. (It is sometimes handy to keep track of whether something is oriented or pseudo-oriented, but you can ignore the difference in calculations as long as you always use the right-hand rule.) The unoriented forms are least used in applications; they typically arise by taking the absolute value of another form (and then possibly multiplying by a scalar field). However, they are still important, since lengths, areas, and volumes may be found by integrating unoriented forms. (If you read other material on differential forms, the oriented ones are the default, and people will leave off the word 'oriented'. Then the pseudoriented forms are just called 'pseudo-forms', and there is no common name for the unoriented forms at all! That said, there are yet other kinds of differential forms besides all of these.)

We'll integrate these forms along various regions in space, called *manifolds*. These manifolds can also be oriented, pseudo-oriented, or unoriented. Now it's the unoriented manifolds that are the simplest; they are just shapes of consistent dimension. With an oriented manifold, we also make a choice of which direction to go along the manifold; with a pseudo-oriented manifold, we instead make a choice of which direction to go around or across the manifold. As you might guess, we integrate oriented forms on oriented manifolds, pseudo-oriented forms on pseudo-oriented manifolds, and unoriented forms on unoriented manifolds. (If you read other material, the pseudo-oriented manifolds are sometimes also called 'transversely oriented'. People will also talk about integrating on *chains*: a chain is just a list of manifolds, each with a real number; to integrate a differential form on a chain, you multiply the integral on each manifold by the corresponding real number and then add these products.)

To calculate integrals, we will parametrize our manifolds; we'll have one or more variables $t, u, v, \ldots$ (the *parameters*), taking a bounded domain of values, and a function (the *parametrization*) specifying which point in space corresponds to which values of the parameters. Running this function over the entire domain of parameters carves out the manifold. (We'll want our parametrization functions to be continuously differentiable, in order to avoid technicalities about whether the integrals are defined. For the same reason, the forms themselves should be continuous, and the domains of the paremetrizations should be closed and bounded. The integrals may be defined in any case, but they are guaranteed to exist if these conditions are met.)

The number of parameters used is the *dimension* of the manifold. This must match the *rank* of the differential form, which is the number of differentials in each term of the form. These differentials are combined using the *wedge product*, $\wedge$. A key property of the wedge product is that it is *anticommutative* between differentials; that is,

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$$

(much like the cross product of vectors). This also means that $\mathrm{d}x \wedge \mathrm{d}x = 0$. However, for unoriented forms, we take the absolute value of the wedge product; then $|\mathrm{d}x \wedge \mathrm{d}y| = |-\mathrm{d}y \wedge \mathrm{d}x| = |\mathrm{d}y \wedge \mathrm{d}x|$, while $|\mathrm{d}x \wedge \mathrm{d}x| = |0| = 0$ still.

To calculate the integral, you use the parametrization to express the coordinates $x, y, \ldots$ in terms of the parameters $t, u, v, \ldots$, then differentiate this to get $\mathrm{d}x, \mathrm{d}y, \ldots$ in terms of $\mathrm{d}t, \mathrm{d}u, \mathrm{d}v, \ldots$, so that the integral is entirely in terms of the parameters. We then express this as an iterated integral, making sure that the (pseudo)-orientation (if any) matches (or else putting a minus sign out front if it doesn't).

Curves

A **curve** C is a manifold of dimension 1. So it is given by a function taking one variable t to a point $R = (x, y, \ldots)$ (which function we'll assume is continuously differentiable). Note that the differential $\mathrm{d}R = \langle \mathrm{d}x, \mathrm{d}y, \ldots \rangle$ is a vector; if we write $\mathbf{r}$ for the vector $R - (0, 0, \ldots)$, then $\mathrm{d}R = \mathrm{d}\mathbf{r}$, and $\mathrm{d}\mathbf{r}$ is the more usual notation (even though R is the more fundamental concept). When we orient a curve, we specify which direction to travel along the curve; when we pseudo-orient a curve in 2 dimensions, we specify which direction to travel across the curve. (We won't need to pseudo-orient a curve in more dimensions in this class, although it can be done by specifying directions around the curve.)

To integrate a vector field $\mathbf{F} = \langle M, N, \ldots \rangle$ along an oriented curve C, we integrate the rank-1 oriented form $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \langle M, N, \ldots \rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \ldots \rangle = \int_C (M\,\mathrm{d}x + N\,\mathrm{d}y + \cdots) = \int_C \left(M \frac{\mathrm{d}x}{\mathrm{d}t} + N \frac{\mathrm{d}y}{\mathrm{d}t} + \cdots \right) \mathrm{d}t$$

or

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_C \langle M, N, \ldots \rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots \right\rangle \mathrm{d}t = \int_C \left(M \frac{\mathrm{d}x}{\mathrm{d}t} + N \frac{\mathrm{d}y}{\mathrm{d}t} + \ldots \right) \mathrm{d}t.$$

(There's no need to learn all of these formulas; just put everything in terms of t and push through.) To match orientations, make sure that the direction along the curve as t changes is the same direction as the curve's orientation; or if not, then put a minus sign out front.

To integrate a vector field $\mathbf{F} = \langle M, N \rangle$ across a pseudo-oriented curve C in 2 dimensions, we integrate the rank-1 pseudo-oriented form $\mathbf{F} \times \mathrm{d}\mathbf{r}$ (where the cross product in 2 dimensions produces a scalar, or rather a pseudo-scalar since the sign depends on the right-hand rule):

$$\int_C \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_C \langle M, N \rangle \times \langle \mathrm{d}x, \mathrm{d}y \rangle = \int_C (M\,\mathrm{d}y - N\,\mathrm{d}x) = \int_C \left(M \frac{\mathrm{d}y}{\mathrm{d}t} - N \frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t$$

or

$$\int_C \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_C \langle M, N \rangle \times \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t} \right\rangle \mathrm{d}t = \int_C \left(M \frac{\mathrm{d}y}{\mathrm{d}t} - N \frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t.$$

To match pseudo-orientations using the right-hand rule, make sure that the direction along the curve as t changes is counterclockwise from the direction of the curve's pseudo-orientation; or if not, then put a minus sign out front.

To integrate a scalar field f on an unoriented curve C, we integrate the rank-1 unoriented form $f\,\text{đ}s$, where s has no meaning by itself but instead $\text{đ}s$ is the unoriented form $\|\mathrm{d}\mathbf{r}\|$:

$$\int_C f\,\text{đ}s = \int_C f\,\|\mathrm{d}\mathbf{r}\| = \int_C f\,\|\langle \mathrm{d}x, \mathrm{d}y, \ldots \rangle\| = \int_C f \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + \cdots} = \int_C f \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \cdots}\,|\mathrm{d}t|$$

or

$$\int_C f\,\text{đ}s = \int_C f\,\|\mathrm{d}\mathbf{r}\| = \int_C f \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| |\mathrm{d}t| = \int_C f \left\| \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots \right\rangle \right\| |\mathrm{d}t| = \int_C f \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \cdots}\,|\mathrm{d}t|.$$

Now there is no orientation to match; instead, make sure that t is increasing, so that $|\mathrm{d}t| = \mathrm{d}t$ in the integral; or if not, then put a minus sign out front.

Surfaces

A **surface** R is a manifold of dimension 2, given by a function taking two variables u, v to a point $R = (x, y, z, \ldots)$ (which function we'll assume is continuously differentiable). When we pseudo-orient a surface in 3 dimensions, we specify which direction to travel across the surface. (We won't need to pseudo-orient a surface in more dimensions, nor will we orient any at all, although again these can be done.)

To integrate a vector field $\mathbf{F} = \langle M, N, O\rangle$ across a pseudo-oriented surface R in 3 dimensions, we integrate the rank-2 pseudo-oriented form $\mathbf{F} \cdot \text{đ}\mathbf{S}$, where $\mathbf{S}$ has no meaning by itself, but instead $\text{đ}\mathbf{S}$ is the psuedo-vector-valued form $1/2\,\mathrm{d}\mathbf{r} \hat{\times} \mathrm{d}\mathbf{r}$ (which as a vector is multiplied by the cross product and as a differential form is multiplied by the wedge product). This works out to $\langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y\rangle$ (using the right-hand rule) or $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\,\mathrm{d}u \wedge \mathrm{d}v$:

$$\int_R \mathbf{F} \cdot \text{đ}\mathbf{S} = \int_R \langle M, N, O\rangle \cdot \langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y\rangle = \int_R (M\,\mathrm{d}y \wedge \mathrm{d}z + N\,\mathrm{d}z \wedge \mathrm{d}x + O\,\mathrm{d}x \wedge \mathrm{d}y)$$
$$= \int_R \left(M\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + N\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + O\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\right) \mathrm{d}u \wedge \mathrm{d}v$$

or

$$\int_R \mathbf{F} \cdot \text{đ}\mathbf{S} = \int_R \langle M, N, O\rangle \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u \wedge \mathrm{d}v = \int_R \langle M, N, O\rangle \cdot \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle \mathrm{d}u \wedge \mathrm{d}v$$
$$= \int_R \left(M\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + N\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + O\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\right) \mathrm{d}u \wedge \mathrm{d}v.$$

To match pseudo-orientations using the right-hand rule, make sure that, as you curl from the direction in which u changes towards the direction in which v changes, the right-hand rule gives the direction of the surface's pseudo-orientation; or if not, then put a minus sign out front.

To integrate a scalar field f on an unoriented surface R, we integrate the rank-2 unoriented form $f\,\text{đ}\sigma$, where σ has no meaning by itself but instead $\text{đ}\sigma$ is the unoriented form $\|\text{đ}\mathbf{S}\|$:

$$\int_R f\,\text{đ}\sigma = \int_R f\,\|\text{đ}\mathbf{S}\| = \int_R f\,\|\langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y\rangle\| = \int_R f\sqrt{(\mathrm{d}y \wedge \mathrm{d}z)^2 + (\mathrm{d}z \wedge \mathrm{d}x)^2 + (\mathrm{d}x \wedge \mathrm{d}y)^2}$$
$$= \int_R f\sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}\;|\mathrm{d}u \wedge \mathrm{d}v|$$

or

$$\int_R f\,\text{đ}\sigma = \int_R f\,\|\text{đ}\mathbf{S}\| = \int_R f\left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| |\mathrm{d}u \wedge \mathrm{d}v| = \int_R f\left\|\left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle\right\| |\mathrm{d}u \wedge \mathrm{d}v|$$
$$= \int_R f\sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}\;|\mathrm{d}u \wedge \mathrm{d}v|.$$

Again there is no orientation to match; instead, make sure that u and v are both increasing, so that $|\mathrm{d}u \wedge \mathrm{d}v| = \mathrm{d}u\,\mathrm{d}v$ in the integral; or if not, then put a minus sign out front for each one that doesn't.

The Stokes Theorem

The (second) Fundamental Theorem of Calculus states that

$$\int_a^b \mathrm{d}f = f|_a^b.$$

This works just as well when there are several independent variables as when there is just one. In this case, we can also write $\mathrm{d}f$ as $\nabla f \cdot \mathrm{d}\mathbf{r}$ to get the theorem

$$\int_a^b \nabla f \cdot \mathrm{d}\mathbf{r} = f|_a^b.$$

Although this is now a theorem about integrating a gradient along a curve, in essence it is still just the FTC, a theorem about integrating differentials.

This theorem generalizes to differential forms of higher rank, where it is called the **Stokes Theorem**:

$$\int_M \mathrm{d} \wedge \alpha = \int_{\partial M} \alpha.$$

Here, α is any oriented or pseudo-oriented differential form and M is any oriented or pseudo-oriented manifold, so long as they have the same kind of orientation and the dimension of M is 1 more than the rank of α (so that the dimensions and ranks in each integral match up). To do this properly, we need to know two things: how to take the differential of a differential form (which is the $\mathrm{d} \wedge \alpha$ in the Stokes Theorem), and how to take the endpoints of a manifold other than a curve (which is the ∂M in the Stokes Theorem).

With endpoints, we're really dealing with the *boundary* of a manifold. The boundary of a curve oriented from a to b consists of both $\{a\}$ and $\{b\}$, the former negatively and the latter positively. (Technically, this is a chain: the point $\{a\}$ has weight -1, while the point $\{b\}$ has weight 1.) If you think of a point $\{a\}$ as a manifold of dimension 0 and think of a scalar quantity f as a differential form of rank 0, then we integrate f on $\{a\}$ by simply taking the value of f at a: $\int_{\{a\}} f = f|_a$, so $\int_{-1\{a\}+1\{b\}} f = -1f|_a + 1f|_b = f|_a^b$. Then the FTC can be written as

$$\int_C \mathrm{d}f = \int_{\partial C} f,$$

so the symbol '∂' indicates the boundary here. The boundary of a surface is a curve (or a chain made up of several curves), and the boundary of a region of space is a surface (or a chain made up of several surfaces).

When we take the differential of a differential form α, we get another differential form if we take the *exterior* differential $\mathrm{d} \wedge \alpha$ (which is usually written just '$\mathrm{d}\alpha$', although there are other kinds of differentials that this could mean instead). When we add forms, the exterior differential obeys the Sum Rule as usual; when we multiply them, we have a kind of Product Rule too. This is the same as the usual Product Rule, except that we must keep track of the order of multiplication. However, this caveat really doesn't matter due to the next rule: the exterior differential of a differential is zero. For example,

$$\mathrm{d} \wedge (x\,\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + x\,\mathrm{d} \wedge \mathrm{d}y \wedge \mathrm{d}z - x\,\mathrm{d}y \wedge \mathrm{d} \wedge \mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + 0 + 0 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

So in the end, you just take the differential of the non-differential portion of each term, then stick this with a wedge in front of the previous differential portion.

When we relate differential forms to vector fields, we'll also use various ways of taking derivatives of vector fields. These can be expressed using ∇ and one of the ways of multiplying vectors: the **divergence** $\nabla \cdot \mathbf{F}$ is a scalar field, and the **curl** $\nabla \times \mathbf{F}$ is a pseudo-vector field in 3 dimensions or a pseudo-scalar field in 2 dimensions. Specifically,

$$\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \ldots \rangle \cdot \langle M, N, \ldots \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \cdots;$$

and

$$\nabla \times \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \times \langle M, N, O \rangle = \left\langle \frac{\partial O}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial O}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

in 3 dimensions, while

$$\nabla \times \mathbf{F} = \langle \partial/\partial x, \partial/\partial y \rangle \times \langle M, N \rangle = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

in 2 dimensions.

The connection between these and differentials is as follows:

- $\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r}$ in any number of dimensions;
- $\mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla \times \mathbf{F}\,\text{đ}A$ in 2 dimensions;
- $\mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla \times \mathbf{F} \cdot \text{đ}\mathbf{S}$ in 3 dimensions;
- $\mathrm{d} \wedge (\mathbf{F} \times \mathrm{d}\mathbf{r}) = \nabla \cdot \mathbf{F}\,\text{đ}A$ in 2 dimensions; and
- $\mathrm{d} \wedge (\mathbf{F} \cdot \text{đ}\mathbf{S}) = \nabla \cdot \mathbf{F}\,\text{đ}V$ in 3 dimensions.

(These are not new principles, but rather facts that you can verify by writing everything in terms of the components of $\mathbf{F}$, partial deriviatves, and differentials.) Here, $\text{đ}A$ is the trivially pseudo-oriented area form $|\mathrm{d}x \wedge \mathrm{d}y|$ (which we identify with $\mathrm{d}x \wedge \mathrm{d}y$ using the right-hand rule), and $\text{đ}V$ is the trivially pseudo-oriented volume form $|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z|$ (which we identify with $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ using the right-hand rule).

Now suppose that a surface R is bounded by a curve ∂R. The Stokes Theorem tells us that

$$\int_R \mathrm{d} \wedge \alpha = \int_{\partial R} \alpha,$$

where α is any (oriented or pseudo-oriented) differential form of rank 1. If I integrate a vector field $\mathbf{F}$ along ∂R, then I'm really integrating the differential form $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$, so

$$\int_{\partial R} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_R \mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \int_R \nabla \times \mathbf{F} \cdot \text{đ}\mathbf{S}$$

in 3 dimensions, or

$$\int_{\partial R} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_R \mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \int_R \nabla \times \mathbf{F} \, \text{đ}A$$

in 2 dimensions. These are the theorems traditionally called *Stokes's Theorem* and *Green's Theorem*, respectively. If, in 2 dimensions, I integrate $\mathbf{F}$ across ∂R, then

$$\int_{\partial R} \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_R \mathrm{d} \wedge (\mathbf{F} \times \mathrm{d}\mathbf{r}) = \int_R \nabla \cdot \mathbf{F} \, \text{đ}A,$$

which is another form of Green's Theorem; in terms of differentials, it's just like the previous version, except that the form being integrated is pseudo-oriented instead of oriented. (These theorems are not new principles either, but follow from the general Stokes Theorem and the differentials on the bottom of the previous page.)

Next, suppose that a bounded region Q in space is bounded by a surface ∂Q. Now the Stokes Theorem tells us that

$$\int_Q \mathrm{d} \wedge \alpha = \int_{\partial Q} \alpha,$$

where now α is any (oriented or pseudo-oriented) differential form of rank 2. If I integrate a vector field $\mathbf{F}$ across ∂Q, then I'm really integrating $\mathbf{F} \cdot \text{đ}\mathbf{S}$, so

$$\int_{\partial Q} \mathbf{F} \cdot \text{đ}\mathbf{S} = \int_Q \mathrm{d} \wedge (\mathbf{F} \cdot \text{đ}\mathbf{S}) = \int_Q \nabla \cdot \mathbf{F} \, \text{đ}V.$$

This is the theorem traditionally called *Gauss's Theorem*, although many textbooks simply call it the *Divergence Theorem*. (Once more, you can verify these by explicit calculation.)

Since the boundary ∂M for any manifold is closed in on itself, the boundary of the boundary, $\partial\partial M$, is always empty. This means that

$$\int_M \mathrm{d} \wedge \mathrm{d} \wedge \alpha = \int_{\partial M} \mathrm{d} \wedge \alpha = \int_{\partial\partial M} \alpha = 0;$$

since this is true no matter how small M may be, we can conclude that

$$\mathrm{d} \wedge \mathrm{d} \wedge \alpha = 0$$

for any (oriented or pseudo-oriented) differential form α. In terms of vector fields, this has two consequences:

$$\nabla \times \nabla f = 0$$

in 2 or 3 dimensions, and

$$\nabla \cdot \nabla \times \mathbf{F} = 0$$

in 3 dimensions. If you write these facts out using partial derivatives, then you'll see that they simply state the equality of mixed partial deriviatives. (As a technicality, that equality is only guaranteed when the mixed partial derivatives are continuous; we derived these facts by considering integrals that likewise are only guaranteed to exist when the forms being integrated are continuous. Conversely, the Stokes Theorem can be proved in the first place by using the equality of mixed partial derivatives and carefully keeping track of everything.)

Hodge duals

You may notice that a vector field $\mathbf{F}$ can be turned into a differential form in two different ways. In 2 dimensions, $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is an oriented form of rank 1, while $\mathbf{F} \times \mathrm{d}\mathbf{r}$ is a pseudo-oriented form of rank 1. In 3 dimensions, $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is again an oriented form of rank 1, while now $\mathbf{F} \cdot đ\mathbf{S}$ is a pseudo-oriented form of rank 2. Either way, the two differential forms related to a single vector field are called *Hodge duals* of each other. People who work directly with differential forms use the Hodge duals to bring in geometric ideas of length and angle, without ever going through vector fields. In this way, one can work as much as possible directly with the objects that one integrates to get measurable quantities.

The Hodge dual of a differential form α is denoted $*\alpha$. In rectangular coordinates, it's easy to calculate Hodge duals; you change the differential part of each term to whatever is missing in the area or volume form (written in the order given by the right-hand rule), paying attention to the sign. This gives us

$$*\mathrm{d}x = \mathrm{d}y,\ *\mathrm{d}y = -\mathrm{d}x$$

in 2 dimensions; and

$$*\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z,\ *\mathrm{d}y = -\mathrm{d}x \wedge \mathrm{d}z = \mathrm{d}z \wedge \mathrm{d}x,\ *\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y$$

and

$$*(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x,\ *(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}y,\ *(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z$$

in 3 dimensions. (The Hodge dual of an oriented form is pseudo-oriented and vice versa, and these rules are written using the right-hand rule.) Now you can check that

$$*(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \mathbf{F} \times \mathrm{d}\mathbf{r},\ *(\mathbf{F} \times \mathrm{d}\mathbf{r}) = -\mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

in 2 dimensions; and

$$*(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \mathbf{F} \cdot đ\mathbf{S},\ *(\mathbf{F} \cdot đ\mathbf{S}) = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

in 3 dimensions. We can even extend this to forms of top rank and to scalar functions (which are differential forms of rank 0):

$$*(đA) = *(\mathrm{d}x \wedge \mathrm{d}y) = 1,\ *1 = \mathrm{d}x \wedge \mathrm{d}y = đA$$

in 2 dimensions; and

$$*(đV) = *(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = 1,\ *1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = đV$$

in 3 dimensions.

Laplacians

The **Laplacian** of a form α is

$$\Delta\alpha = *\big(\mathrm{d} \wedge *(\mathrm{d} \wedge \alpha)\big) \pm \mathrm{d} \wedge *(\mathrm{d} \wedge *\alpha),$$

where we use $+$ or $-$ on the second term depending on whether the ambient space has even or odd dimension, and we throw in another overall minus sign iff both the space's dimension and the form's rank are odd. (I know, that's kind of a complicated rule. Just be glad that I'm not trying to do this in non-Euclidean spaces too.) In other words, take the exterior differential, then the Hodge dual, then repeat; and also do this in reverse order; then add or subtract these according to the parity of the dimension, and possibly take the opposite of the entire result. Notice that $\Delta\alpha$ has both the same rank and the same orientation as α, so it is a nice notion of second derivative.

If we think of a scalar field f as an oriented form of rank 0, then $\mathrm{d} \wedge f = \mathrm{d}f$, while $*f$ has top rank, so $\mathrm{d} \wedge *f = 0$. Then

$$\Delta f = *(\mathrm{d} \wedge *\mathrm{d}f) = *\big(\mathrm{d} \wedge *(\nabla f \cdot \mathrm{d}\mathbf{r})\big) = *\big(\mathrm{d} \wedge (\nabla f \times \mathrm{d}\mathbf{r})\big) = *(\nabla \cdot \nabla f\, đA) = \nabla \cdot \nabla f$$

in 2 dimensions; and

$$\Delta f = *(\mathrm{d} \wedge *\mathrm{d}f) = *\Big(\mathrm{d} \wedge *(\nabla f \cdot \mathrm{d}\mathbf{r})\Big) = *\Big(\mathrm{d} \wedge (\nabla f \cdot \text{đ}\mathbf{S})\Big) = *(\nabla \cdot \nabla f \, \text{đ}V) = \nabla \cdot \nabla f$$

in 3 dimensions. In fact, the rule that $\Delta f = \nabla \cdot \nabla f$ is correct in *any* number of dimensions (and the weird rules about minus signs are designed to make that work out); for this reason, the Laplacian operator Δ is often written as '$\|\nabla\|^2$' or just '∇^2' (think of $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$).

Other Laplacians are

$$\Delta(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla(\nabla \cdot \mathbf{F}) \cdot \mathrm{d}\mathbf{r} + \nabla(\nabla \times \mathbf{F}) \times \mathrm{d}\mathbf{r},\ \Delta(\mathbf{F} \times \mathrm{d}\mathbf{r}) = \nabla(\nabla \cdot \mathbf{F}) \times \mathrm{d}\mathbf{r} - \nabla(\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{r}$$

in 2 dimensions; and

$$\Delta(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla(\nabla \cdot \mathbf{F}) \cdot \mathrm{d}\mathbf{r} - \nabla \times (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{r},\ \Delta(\mathbf{F} \cdot \text{đ}\mathbf{S}) = \nabla(\nabla \cdot \mathbf{F}) \cdot \text{đ}\mathbf{S} - \nabla \times (\nabla \times \mathbf{F}) \cdot \text{đ}\mathbf{S}$$

in 3 dimensions. If you define $\Delta\mathbf{F}$ so that $\Delta\mathbf{F} \cdot \mathrm{d}\mathbf{r} = \Delta(\mathbf{F} \cdot \mathbf{r})$, you can see that $\Delta\mathbf{F} \times \mathrm{d}\mathbf{r} = \Delta(\mathbf{F} \times \mathrm{d}\mathbf{r})$ in 2 dimensions and that $\Delta\mathbf{F} \times \mathrm{d}\mathbf{S} = \Delta(\mathbf{F} \times \mathrm{d}\mathbf{S})$ in 3 dimensions; furthermore, each component of $\Delta\mathbf{F}$ is the Laplacian of the corresponding component of $\mathbf{F}$. So Laplacians work very nicely indeed.