The Fundamental Theorem of Calculus

In one-variable Calculus, the second Fundamental Theorem states that

$$\int_{x=a}^{b} f'(x)\,\mathrm{d}x = f(b) - f(a).$$

If we write u for the quantity $f(x)$, then its differential $\mathrm{d}u$ is precisely the integrand $f'(x)\,\mathrm{d}x$, so the Fundamental Theorem can also be written as

$$\int_a^b \mathrm{d}u = u\big|_a^b.$$

This works just as well when there are several independent variables as when there is just one. Now if $u = f(P)$, then $\mathrm{d}u$ is $\nabla f(P) \cdot \mathrm{d}\mathbf{r}$, so

$$\int_{P=a}^{b} \nabla f(P) \cdot \mathrm{d}\mathbf{r} = f(b) - f(a).$$

Although this is now a theorem about integrating a gradient along a curve, in essence it is still just the FTC, a theorem about integrating differentials. This has a massive generalization to higher-rank differential forms, called the *Stokes Theorem*, which we'll get to later.

A differential form is called **exact** if there exists a quantity u such that $\alpha = \mathrm{d}u$. Similarly, a vector field $\mathbf{F}$ is called **conservative** if there is a scalar field f such that $\mathbf{F} = \nabla f$. The connection between these is that $\mathbf{F}$ is conservative if and only if $\mathbf{F}(P) \cdot \mathrm{d}\mathbf{r}$ is exact. (After all, if $\mathbf{F} = \nabla f$, then $\mathbf{F}(P) \cdot \mathrm{d}\mathbf{r} = \mathrm{d}\big(f(P)\big)$.) An oriented curve is called **closed** if its beginning and ending points are the same; one sometimes emphasizes that an integral is along a closed curve by writing $\oint$ in place of $\int$. Then the integral of an exact differential form or a conservative vector field along a closed curve is zero, because

$$\oint_C \alpha = \int_a^a \mathrm{d}u = u\big|_a^a = u\big|_a - u\big|_a = 0.$$

Similarly, the integral of a conservative vector field along a closed curve is zero. In this case, we can use notation more like that of a definite integral in one variable:

$$\int_{P=P_1}^{P_2} \alpha$$

means the integral of α along *any* curve from P_1 to P_2. It doesn't matter which curve you use; if C_1 and C_2 are both curves like this, then these combine into a closed curve $C_1 - C_2$, in which you start at P_1, follow C_1 to P_2, then follow C_2 backwards (hence the minus sign) back to P_1. Then

$$\int_{C_1} \alpha - \int_{C_2} \alpha = \oint_{C_1 - C_2} \alpha = 0,$$

so $\int_{C_1} \alpha = \int_{C_2} \alpha$. (This is still undefined if there is *no* curve from P_1 to P_2 through the domain of α. This is analogous to the case in one dimension of an integral $\int_{x=a}^{b} f(x)\,\mathrm{d}x$ where f is undefined somewhere between a and b.)

Conversely, if the integral of a differential form or of a vector field is zero along *every* closed curve, then that differential form must be exact or that vector field must be conservative. The reason is that in this case (and only in this case) we can pick a point P_0 to start from and define a semidefinite integral

$$u = \int_{P=P_0} \alpha = \int_{P_0}^{P} \alpha.$$

Because α is exact, you get the same result no matter which path you use from P_0 to P. (Ideally, the domain of α should be *path-connected*, meaning that there exists a curve between any two points. If not, then you must split the domain into various path-connected components and pick a point in each.) That $\mathrm{d}u = \alpha$ in this case is essentially the multivariable version of the *first* Fundamental Theorem of Calculus.

Given a differential form α, finding such an expression u is a form of *indefinite* integration. It's not practical to check every possible curve, of course, so we need other methods to decide if α is exact, and this can also help us to find u. There are actually several methods; one is given in the textbook, essentially reversing the process of partial differentiation with a kind of partial integration. (If you try this method when α is not exact, then it will fail.)

If the domain of α is reasonably simple, then it's possible to pick a point P_0 and write down a general formula for a parametrized curve from P_0 to any point P. (For example, you could always use a straight line segment, as long as these line segments always lie entirely within the domain.) If you try this method when α is not exact, then you may get a result; but when you check it, then you'll find that it's wrong when α is not exact.

It's often possible to tell ahead of time whether α is exact. To really explain what's going on here, I'll need to talk about the *exterior differential*, which is a topic that we'll get to in a couple of weeks. For now, I'll describe it in terms of partial derivatives. So, if $\alpha = \mathrm{d}u$, then

$$\alpha = \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y + \cdots.$$

(The dots are meant to indicate that more terms may appear if there are more than two variables.) Assuming that u is twice differentiable, then mixed second partial derivatives are equal:

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

So if you start with an arbitrary linear differential 1-form

$$\alpha = \alpha_x\,\mathrm{d}x + \alpha_y\,\mathrm{d}y + \cdots,$$

then it could only be exact if it is **closed**, meaning that

$$\frac{\partial \alpha_x}{\partial y} = \frac{\partial \alpha_y}{\partial x}$$

(and similarly for other mixtures of derivatives if there are more than two variables), assuming that it's differentiable in the first place. Similarly, a vector field

$$\mathbf{F}(x, y, \ldots) = \mathbf{F}_1(x, y, \ldots)\mathbf{i} + \mathbf{F}_2(x, y, \ldots)\mathbf{j} + \cdots$$

can only be conservative if it is **irrotational**, meaning that

$$D_2\mathbf{F}_1 = D_1\mathbf{F}_2$$

(and similarly for other mixtures of derivatives if there are more than two variables), assuming that it's differentiable in the first place.

Conversely, a closed differential form or an irrotational vector field must be exact or conservative (respectively) if its domain is **precisely-simply connected**, which means that any simple closed curve (one that doesn't intersect itself except where its two endpoints are equal) in the domain of the differential form or the vector field is the boundary of a region that lies entirely within that domain. (The domain is *simply connected* if it is both path-connected and precisely-simply connected. Conversely, it is precisely-simply connected if each of its path-connected components is simply connected. If you take a class in Topology such as MATH 471 at UNL, then you'll learn a hundred specific terms like these.) But a full discussion of the reasons for this must wait until we've covered higher-order differential forms.