Point- and vector-valued functions

The book often doesn't distinguish between a point P and its position vector $\mathbf{r} = P - O$, where O is the origin of a coordinate system. Conceptually, they're very different, since you can talk about points and vectors geometrically without bringing coordinates into it, so the concepts are meaningful even if there is no such thing as O to equivocate them. On the other hand, when doing calculations, it's easy to conflate them; since the coordinates of O are all 0, when you do the subtraction, the components of $\mathbf{r}$ are exactly the same as the coordinates of P. Still, you should always keep in mind whether a given expression really refers to a point or to a vector.

If P is a point, then the difference ΔP is a vector (because it's the result of subtracting two points), and then the differential $\mathrm{d}P$ is an infinitesimal vector. If P is a function of some scalar quantity t, then $\mathrm{d}P/\mathrm{d}t$ makes sense, because it's a vector divided by a scalar, but now it's no longer infinitesimal (unless it happens to be zero). In other words, *the derivative of a point with respect to a scalar is a vector*. Another way to say this is that if f is a point-valued function, then its derivative f' is a vector-valued function:

$$f'(t) = \lim_{h \to 0} \left(\frac{f(t+h) - f(t)}{h} \right);$$

first subtract two points to get a vector, then divide by the scalar h to get another vector, and finally take the limit of these vectors to get a vector. Of course, the derivative of a *vector* with respect to a scalar is *also* a vector; in other words, the derivative of a vector-valued function is also a vector-valued function.

For example, if P gives the position of some object at time t, then P is a point, but $\mathrm{d}P/\mathrm{d}t$, the *velocity* of the object, is a vector. (Note that the magnitude of this vector is the object's *speed*.) If we write $\mathbf{v}$ for $\mathrm{d}P/\mathrm{d}t$, then $\mathrm{d}\mathbf{v}/\mathrm{d}t$ is the acceleration of the object, which is also a vector. (Physicists and mechanical engineers use the word 'acceleration' like this, to indicate any change in velocity —speed or direction— over time. In everyday language, this word means something more like $\mathrm{d}\,|\mathbf{v}|/\mathrm{d}t$, the derivative of speed with respect to time, which is the same as the scalar component of the acceleration in the direction of the velocity. This is positive if the object is speeding up and negative if the object is slowing down, or decelerating.)

Reversing this, if you take the indefinite integral of a vector, then the result may be either a point *or* a vector, because differentiating either of these yields a vector. This ambiguity is packaged into the constant of integration. For example, $\int \langle 2t, 3\rangle \,\mathrm{d}t = \langle t^2, 3t\rangle + C$, which is a point if C is a point and a vector if C is a vector. The definite integral of a vector is always a vector: fundamentally, you get it by adding up infinitely many infinitesimal vectors (or approximate it by adding up a large number of small vectors), and adding up vectors yields a vector; in practice, you usually calculate it by subtracting indefinite integrals, and regardless of whether you view the indefinite integrals as points or as vectors, subtracting them yields a vector. For example, both $\int_{t=0}^{1} \langle 2t, 3\rangle \,\mathrm{d}t = \langle t^2, 3t\rangle\big|_{t=0}^{1} = \langle 1, 3\rangle - \langle 0, 0\rangle = \langle 1, 3\rangle$, and $\int_{t=0}^{1} \langle 2t, 3\rangle \,\mathrm{d}t = (t^2, 3t)\big|_{t=0}^{1} = (1, 3) - (0, 0) = \langle 1, 3\rangle$ give the same result.

Putting this all together, consider the initial-value problem in which the acceleration of an object is $-32\mathbf{k} = \langle 0, 0, -32\rangle$ (which is the acceleration of a freely falling object near Earth's surface, if we use units of feet and seconds), the object's initial velocity is $\langle 3, 0, 4\rangle$ (so a speed of $5\,\mathrm{ft/s}$ eastward and upward with a slope of $4/3$), and the object's initial position is $(0, 0, 100)$ (so 100 feet above the origin on the ground). Then you can calculate a general formula for the object's position P as a function of the elapsed time t by integrating:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \langle 0, 0, -32 \rangle; \\
\mathrm{d}\mathbf{v} &= \langle 0, 0, -32 \rangle \, \mathrm{d}t; \\
\int_{\mathbf{v}=\langle 3,0,4 \rangle} \mathrm{d}\mathbf{v} &= \int_{t=0} \langle 0, 0, -32 \rangle \, \mathrm{d}t; \\
\mathbf{v} - \langle 3, 0, 4 \rangle &= \langle 0, 0, -32t \rangle - \langle 0, 0, -32(0) \rangle; \\
\mathbf{v} &= \langle 3, 0, 4 \rangle + \langle 0, 0, -32t \rangle; \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \langle 3, 0, 4 - 32t \rangle; \\
\mathrm{d}P &= \langle 3, 0, 4 - 32t \rangle \, \mathrm{d}t; \\
\int_{P=(0,0,100)} \mathrm{d}P &= \int_{t=0} \langle 3, 0, 4 - 32t \rangle \, \mathrm{d}t; \\
P - (0, 0, 100) &= \langle 3t, 0, 4t - 16t^2 \rangle - \left\langle 3(0), 0, 4(0) - 16(0)^2 \right\rangle; \\
P &= (0, 0, 100) + \langle 3t, 0, 4t - 16t^2 \rangle; \\
P &= (3t, 0, 100 + 4t - 16t^2).
\end{aligned}$$

In other words, the position after t seconds is $3t$ feet east of the origin at a height of $100 + 4t - 16t^2$ feet.

In the course of solving this, I've used the *semidefinite integral*:

$$\int_{t=a} f(t) \, \mathrm{d}t = \int_{\tau=a}^{t} f(\tau) \, \mathrm{d}\tau.$$

The Fundamental Theorem of Calculus allows us to calculate these integrals easily:

$$\int_{t=a} F'(t) \, \mathrm{d}t = F(t) - F(a).$$

This is very handy when solving initial-value problems. Since $\mathbf{v} = \langle 3, 0, 4 \rangle$ when $t = 0$, the first step in which I introduced integrals is doing the same operation to both sides of the equation; similarly, the second introduction of integrals is valid because $P = (0, 0, 100)$ when $t = 0$. To solve this problem using indefinite integrals instead requires two extra steps (one for each integral) to find the constants of integration, but using semidefinite integrals avoids that.