Differential forms

Differential forms are, broadly speaking, expressions that may have *differentials* in them. They have many uses in modern science and engineering, even though they are not traditionally covered explicitly in math class. They are covered somewhat, however, and they are there whenever you differentiate or integrate, even if you don't recognize them. They are especially prominent in multivariable Calculus, and I want to bring them to your attention; you'll find that symbols that otherwise seem meaningless or merely mnemonic can be understood literally (sometimes with slight changes) as differential forms.

Examples

The most basic examples of differential forms are differentials such as $\mathrm{d}x$ and $\mathrm{d}y$. In general, if u is any quantity that might change, then $\mathrm{d}u$ is intended to be a related quantity whose value is an infinitely small change in u, or rather the amount by which the value of u changes when an infinitely small (or arbitrarily small) change is made. (I will make this precise later on.)

Besides the differentials themselves, differential forms can be constructed by applying arithmetic operations, so $\mathrm{d}x + \mathrm{d}y$, $\mathrm{d}x\,\mathrm{d}y$, and $\sqrt{\mathrm{d}x}$ are all differential forms. In all of these expressions, we adopt an order of operations in which the differential operator d is applied before any arithmetic operator; for example, $\mathrm{d}x^2$ means $(\mathrm{d}x)^2$, not $\mathrm{d}(x^2)$ (which is $\mathrm{d}u$ when $u = x^2$ and happens to be equal to $2x\,\mathrm{d}x$). Additionally, we can include ordinary quantities in these expressions, so $x + \mathrm{d}x$, $3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$, and $x \ln{(y/\mathrm{d}z)}$ are also differential forms. We can also use differentials of differentials, such as d^2x (which means $\mathrm{d}(\mathrm{d}x)$, the differential of $\mathrm{d}x$), although we won't need such *higher-order differentials* in this course. Besides all of this, any ordinary expression counts as a differential form in a degenerate way; thus, x, y^2, and $2xy^3$ are also differential forms (of order zero).

Some differential forms are more useful than others. Of those listed above, besides the differentials and the non-differential quantities, the ones most likely to appear in a real problem are $\mathrm{d}x + \mathrm{d}y$ and $3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$. These consist of any number of terms, each of which is the product of an ordinary quantity (possibly the constant 1) and the differential of an ordinary quantity. Differential forms with this property are most commonly found in practice. We will use other differential forms, such as $3x\,|\mathrm{d}y|$ and $\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$; however, you might be able to see how even these forms are differential of *degree* 1 in a sense similar to the degree of a polynomial.

All of the examples so far are differential forms of *rank* 1; there are also differential forms of higher rank, such as $\mathrm{d}x \wedge \mathrm{d}y$, which are written using a new operation, the *wedge product*. (Since the wedge product is a kind of multiplication, this example is not only rank 2 but also degree 2.) We will not use these until later; these notes are only about differential forms of rank 1, or 1-forms for short. (Ordinary quantities have rank 0; this is why they are useful despite not having degree 1. In general, the useful differential forms have the same degree as their rank, and people who study differential forms most often study the so-called *exterior* differential forms, for which the degree and rank automatically match. However, there are many differential forms, such as $3x\,|\mathrm{d}y|$, which don't count as exterior forms but which we will still need in this course. That is the main reason why I'm introducing a very general concept of differential form to begin with, even though we really only need a few special cases.)

Evaluating differential forms

In this class, we generally assume that any ordinary quantity (that is any 0-form) is a function of 2 or 3 ordinary variables, $P = (x, y)$ or $P = (x, y, z)$. Thus, we evaluate ordinary quantites (0-forms) by specifying specific values for the variables that comprise P. For example, to evaluate $u = x^2 + xy$ when $x = 2$ and $y = 3$, we may write

$$u|_{P=(2,3)} = (x^2 + xy)|_{(x,y)=(2,3)} = (2)^2 + (2)(3) = 10.$$

To evaluate a differential form (of order 1), we need not only a point (a value of P) but also a vector (a value of $\mathrm{d}P$). So for example, to evaluate $\alpha = 3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$ when $x = 2$, $y = 3$, $z = 4$, $\mathrm{d}x = 0.05$, $\mathrm{d}y = -0.01$, and $\mathrm{d}z = 0$, we may write

$$\begin{aligned}\alpha|_{\substack{P=(2,3,4),\\ \mathrm{d}P=\langle 0.05,-0.01,0\rangle}} &= (3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z)|_{\substack{(x,y,z)=(2,3,4),\\ \langle \mathrm{d}x,\mathrm{d}y,\mathrm{d}z\rangle=\langle 0.05,-0.01,0\rangle}} \\ &= 3(0.05) + (2)^2(-0.01) + \mathrm{e}^{(3)}(0) = 0.11.\end{aligned}$$

(Differential forms are often denoted with Greek letters such as 'α', although they don't have to be.) We say that α has been evaluated *at* the point $P = (2, 3, 4)$ *along* the vector $\mathrm{d}P = \langle 0.05, -0.01, 0\rangle$. (The components of $\mathrm{d}P$ don't need to be small, since the definition makes sense in any case, but in applications that's usually what matters; after all, $\mathrm{d}P$ is supposed to be a *small* change in position.)

To evaluate higher-order differential forms (those that involve higher-order differentials), we need to specify additional vectors such as $\mathrm{d}^2P = \langle \mathrm{d}^2x, \mathrm{d}^2y, \mathrm{d}^2z\rangle$, etc. However, we won't need that level of generality in this course.

Differential forms as vectors

A differential form $\alpha = M\,\mathrm{d}x + N\,\mathrm{d}y + O\,\mathrm{d}z$ may be viewed as a dot product $\alpha = \langle M, N, O\rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\rangle = \mathbf{V} \cdot \mathrm{d}P$. For example, if $\alpha = 3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$, then $\alpha = \langle 3, x^2, \mathrm{e}^y\rangle \cdot \mathrm{d}P$; conversely, if $\mathbf{V} = \langle 3, x^2, \mathrm{e}^y\rangle$, then

$$\mathbf{V} \cdot \mathrm{d}P = \langle 3, x^2, \mathrm{e}^y\rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\rangle = 3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z.$$

(We can recover $\mathbf{V}$ from α formally by evaluating α when $\mathrm{d}P$ is $\langle \mathbf{i}, \mathbf{j}\rangle$ or $\langle \mathbf{i}, \mathbf{j}, \mathbf{k}\rangle$, but there's probably no need to think about that explicitly.)

Even in circumstances where it makes no sense to interpret a change in the values of (x, y, z) as a vector in the geometric sense (with length and direction), in which case dot products involving them generally have no meaning, it is traditional to write differential forms in this way and to focus on $\mathbf{V}$ rather than on α as the object of study. In this case, we need to think of $\mathbf{V}$ as a *row* vector. Regardless of whether $\mathbf{V}$ has geometric significance as a vector, it can be helpful to visualize it as one.

When calculations with a row vector need to be performed, ultimately it is the differential form $\alpha = \mathbf{V} \cdot \mathrm{d}P$ that matters. It's more common to see $\mathbf{V} \cdot \mathrm{d}\mathbf{r}$, where as usual the vector $\mathbf{r} = P - O$ (where O is $(0, 0)$ or $(0, 0, 0)$) satisfies $\mathrm{d}\mathbf{r} = \mathrm{d}P$. Sometimes $\mathbf{V} \cdot \mathrm{d}\mathbf{r}$ is even regarded as merely a mnemonic notation (especially in the context of defining integrals such as those in Section 15.2 of the textbook), but it can be taken literally, just as $\mathrm{d}y/\mathrm{d}x$ (which is also sometimes regarded as merely mnemonic) can be viewed literally as the result of a division of differentials. In any case, people do write $\mathbf{V} \cdot \mathrm{d}\mathbf{r}$ (even in the textbook), so it can be nice to know what it means!

In the textbook, they also sometimes write $\mathrm{d}\mathbf{r} = \mathbf{T}\,\mathrm{d}s$, where $\mathrm{d}s$ (which is not really the differential of anything) is the magnitude $\mathrm{d}s = |\mathrm{d}\mathbf{r}|$ and $\mathbf{T} = \widehat{\mathrm{d}\mathbf{r}}$, the unit vector in the direction of $\mathrm{d}\mathbf{r}$. This is sometimes useful when thinking about things geometrically, but it's not necessary for purposes of calculation. In 2 dimensions, we can also take cross products (using the rule $\langle a, b\rangle \times \langle c, d\rangle = ad - bc$); for example, if $\mathbf{V} = \langle 3, x^2\rangle$, then

$$\mathbf{V} \times \mathrm{d}\mathbf{r} = \langle 3, x^2\rangle \times \langle \mathrm{d}x, \mathrm{d}y\rangle = 3\,\mathrm{d}y - x^2\,\mathrm{d}x.$$

(This requires that changes in x and y make sense as having a geometric length even when $\mathbf{V}$ is regarded as merely a row vector, so it doesn't come up as often.) If you use $\times\langle c, d\rangle = \langle d, -c\rangle$, so that $\mathbf{u} \times \mathbf{v} = \mathbf{u} \cdot \times\mathbf{v}$, then you can write $\mathbf{V} \times \mathrm{d}\mathbf{r}$ as $\mathbf{V} \cdot \times\mathrm{d}\mathbf{r}$; the book sometimes writes this as $\mathbf{V} \cdot \mathbf{n}\,\mathrm{d}s$, where $\mathrm{d}s = |\times\mathrm{d}\mathbf{r}| = |\mathrm{d}\mathbf{r}|$ again, and now $\mathbf{n} = \widehat{\times\mathrm{d}\mathbf{r}} = \times\mathbf{T}$ is the direction perpendicular and clockwise from $\mathrm{d}\mathbf{r}$. Again, sometimes this is useful when thinking about the geometry, but you don't need it for doing calculations.

This is all especially common when $\mathbf{V}$ is the output of a *vector field*, that is a vector-valued function of several variables. For example, if $\mathbf{F}(x, y) = \langle 3, x^2\rangle$, then $\mathbf{F}(x, y) \cdot \mathrm{d}\mathbf{r} = \langle 3, x^2\rangle \cdot \langle \mathrm{d}x, \mathrm{d}y\rangle = 3\,\mathrm{d}x + x^2\,\mathrm{d}y$, and $\mathbf{F}(x, y) \times \mathrm{d}\mathbf{r} = \langle 3, x^2\rangle \times \langle \mathrm{d}x, \mathrm{d}y\rangle = 3\,\mathrm{d}y - x^2\,\mathrm{d}x$. So in Section 15.2, which is really about integrating differential 1-forms along curves, the book spends most of its time talking about integrating vector fields along curves (and occasionally integrating them across curves in 2 dimensions). What's really going on is that you integrate the vector field $\mathbf{F}$ by integrating one of these two differential forms (usually the first one).