Integration on curves

Differential 1-forms (that is differential forms without the wedge product that we will get to later) can be integrated along curves. To a large extent, that is what they are for. Since differential forms are made of differentials and the definition of the differential of an expression (at least the one that I gave in the handout from January 21) is ultimately about curves, this is a very natural operation.

The definition

Like the textbook does for one-variable Calculus, I'll define the Riemann integral as a limit of Riemann sums, although there are more general notions of integration that can handle more expressions. The Riemann integral will be sufficient for *piecewise continuous* differential forms (those defined in one or more pieces using continuous operations applied to continuous quantities and the differentials of continuously differentiable quantities) along *piecewise continuously differentiable curves* (those with parametrizations defined in one more pieces using continuously differentiable operations applied to the parameter).

So, suppose that we have a differential form α written using the variables $R = (x, y, \ldots)$ and their differentials, and a curve in the same number of dimensions, given by some parametrization function C whose domain is a closed interval $[a, b]$. Then we can try to integrate α along the curve where $R = C(t)$, by defining the integral

$$\int_{R=C(t)} \alpha.$$

Given any way of dividing the interval $[a, b]$ into a partition $a = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = b$ (with n subintervals) and tagging this partition with n values c_k with $t_{k-1} \leq c_k \leq t_k$ for k from 1 to n (this is exactly the kind of partition considered in one-variable Calculus, as on pages 302–304 of the textbook), there is a **Riemann sum**

$$\sum_{k=1}^{n} \alpha\Big|_{\substack{R=C(c_k) \\ \mathrm{d}R=C(t_k)-C(t_{k-1})}}.$$

That is, on the kth subinterval, we evaluate the form α at the point through which the curve passes at time c_k within that subinterval along the vector from where the curve is at the beginning of the subinterval to where it is at the end of the subinterval. If we require that the magnitude of this vector be less than δ and take the limit as $\delta \to 0^+$, then this limit (if it exists) is the value of the integral. And there is a theorem that it does exist, at least if α is piecewise continuous and C is piecewise continuously differentiable (and sometimes otherwise); I don't know a nice proof of this directly, but you can prove that it exists because the practical calculation method on page 2 works.

There is now another nice theorem, that the value of this integral does not depend on the parametrization of the curve, at least not very much. That is, if ϕ is a function in the ordinary sense (a real-valued function of one real variable), then $C \circ \phi$ is another parametrized curve; if ϕ is one-to-one and increasing (so that we travel along the curve in the same direction without repetition) and its range lies entirely within the domain of C (so that we cover the entire curve), then the theorem is that $\int_{R=C(t)} \alpha = \int_{R=(C\circ\phi)(t)} \alpha$. The proof is that any Riemann sum for C is also a Riemann sum for $C \circ \phi$; the same points $C(t_k)$ and $C(c_k)$ occur in the same order, just at different values of the parameter. So the Riemann integrals, which are the limits of these Riemann sums, must also be the same.

For this reason, we usually don't specify a parametrized curve in the notation at all. Instead, we specify an **oriented curve**, which is anything that *could* be given as a parametrized curve, keeping track of which direction we travel along the curve (this is the **orientation** of the curve) but otherwise ignoring the parametrization.

Evaluating integrals on curves

The practical method of evaluating integrals on curves is to pick any convenient parametrization (preferably one that is continuously differentiable) and put everything in terms of that parameter. For example, to integrate $2x\,\mathrm{d}x + 3xy\,\mathrm{d}y$ along the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise, try the parametrization where $x = 2\cos t$, $y = 2\sin t$, and $0 \le t \le \pi$. Then $\mathrm{d}x = -2\sin t\,\mathrm{d}t$ and $\mathrm{d}y = 2\cos t\,\mathrm{d}t$, so the value of the integral is

$$\begin{aligned}\int_{\substack{x^2+y^2=4,\ y\ge 0\\ \mathrm{d}x\le 0}} (2x\,\mathrm{d}x + 3xy\,\mathrm{d}y) &= \int_{t=0}^{\pi} \Big(2(2\cos t)(-2\sin t\,\mathrm{d}t) + 3(2\cos t)(2\sin t)(2\cos t\,\mathrm{d}t)\Big) \\ &= \int_{t=0}^{\pi} (-8\sin t\cos t + 24\sin t\cos^2 t)\,\mathrm{d}t = 16.\end{aligned}$$

(You can do this last integral with the substitution $u = \cos t$.) I've described the curve of integration with an equation (of a circle) and an inequality (to get the top half only) and oriented it by saying that x is always decreasing (so that $\mathrm{d}x$ is always negative), but usually people write that all out to the side somewhere, call the resulting oriented curve C (for example), and write simply $\int_C$.

The reason why this gives the correct result is that any Riemann sum for the integral involving t involves almost the same calculations as a Riemann sum for the integral along the curve. The only difference is that the integral involving t looks at the point from the middle of each subinterval to handle the differentials, whereas as the integral of the curve looks at the points on each end of the subinterval. But in the limit, all of these points approach each other, and the result is the same. (There is another slight complication because the integral involving t takes a limit as the change in t goes to 0, while the integral along the curve takes a limit as the magnitude of the change in position goes to 0. However, these are the same because the parametrization is continuous. If you can calculate $\mathrm{d}x$ and $\mathrm{d}y$ at all, then the parametrization must be differentiable and so definitely continuous.)

You should be able to visualize this example geometrically well enough to see that the answer would have to be positive. The term $2x\,\mathrm{d}x$ should completely cancel, because the right half of the curve exactly mirrors the left half, with $\mathrm{d}x$ the same on both halves (always negative because of movement to the left) but x being the opposite on the two halves (first positive, then negative). On the other hand, the term $3xy\,\mathrm{d}y$ will be negative on both sides; while y is always positive (above the horizontal axis), x and $\mathrm{d}y$ are both positive on the right half (right of the vertical axis and moving upwards) and both negative on the left half (left of the axis and moving downwards), making for a positive product everywhere.

If you are asked to integrate a vector field $\mathbf{F}$ along an oriented curve, then they really want you to integrate the differential form $\mathbf{F}(x,y) \cdot \langle \mathrm{d}x, \mathrm{d}y\rangle$, or more generally $\mathbf{F}(R) \cdot \mathrm{d}R$, where R is (x,y) or (x,y,z). If you write $\mathbf{r}$ for the vector $R - O$ (where O is the origin $(0,0)$ or $(0,0,0)$), then $\mathrm{d}R = \mathrm{d}\mathbf{r}$, and this is the reason for the traditional notation $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, which is used in the textbook. You may also see $\int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s$, where $\mathrm{d}s$ is the $\mathrm{d}s$ that appears at the very bottom of this page and $\mathbf{T}$ is defined to be $\mathrm{d}\mathbf{r}/\mathrm{d}s$. This is usually completely pointless; if you see $\mathbf{T}\,\mathrm{d}s$, just think of it as $\mathrm{d}\mathbf{r}$.

For example, to integrate $\langle 2x, 3xy\rangle$ along the same semicircle as in the previous example (with the same orientation), you do exactly the same integral as in the previous example. This is because

$$\langle 2x, 3xy\rangle \cdot \langle \mathrm{d}x, \mathrm{d}y\rangle = 2x\,\mathrm{d}x + 3xy\,\mathrm{d}y,$$

so

$$\int_C \langle 2x, 3xy\rangle \cdot \mathrm{d}\mathbf{r} = \int_C (2x\,\mathrm{d}x + 3xy\,\mathrm{d}y) = 16$$

as before. Since the vector $\langle 2x, 3xy\rangle$ points to the right on the right side and to the left on the left side, while we move along the curve consistently to the left, this suggests that the horizontal component should cancel. However, since this vector points upwards where we move upwards along the curve (on the right side) and points downwards where we move downards along the curve (on the left side), this suggests a positive contribution from the vertical component. So as in the first example, you should expect a positive result even before doing the calculation.

If you are asked to integrate a function f along a curve, then they really want you to integrate the differential form $f(x,y)\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$, or more generally $f(R)\,|\mathrm{d}R|$. It's traditional to write $\mathrm{d}s$ for $|\mathrm{d}R|$ (or

$|\mathrm{d}\mathbf{r}|$, which is the same), but it's important that there is no quantity s defined everywhere that $\mathrm{d}s$ is the differential of. To emphasize this, you can write $đs$; 'đ' is a symbol that some people use when something is traditionally written with 'd' but is not really a differential.

As long as the differentials $\mathrm{d}x$ etc appear only in $đs$, then the result of the integral is independent of orientation, because replacing $\mathrm{d}x$ with $-\mathrm{d}x$ (as would happen upon reversing the orientation) doesn't change $đs$. For this reason, you can integrate a function along an *unoriented* curve. When parametrizing, everything will come out using $|\mathrm{d}t|$ instead of $\mathrm{d}t$, but as long as the integral involving t has its bounds set up so that t is increasing, then $\mathrm{d}t$ is positive and so $|\mathrm{d}t| = \mathrm{d}t$, after which you can integrate normally.

For example, to find the length of the semicircle in the previous example, you get

$$đs = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{(-2\sin t\,\mathrm{d}t)^2 + (2\cos t\,\mathrm{d}t)^2} = \sqrt{(4\sin^2 t + 4\cos^2 t)\,\mathrm{d}t^2} = \sqrt{4}\sqrt{\mathrm{d}t^2} = 2\,|\mathrm{d}t|.$$

Thus, the length is

$$\int_{x^2+y^2=4,\ y\geq 0} đs = \int_{t=0}^{\pi} 2\,|\mathrm{d}t| = \int_{t=0}^{\pi} 2\,\mathrm{d}t = 2\pi.$$

If for some reason you set the integral up backward, then $\mathrm{d}t$ would be negative and so $|\mathrm{d}t|$ would be $-\mathrm{d}t$, and the result would be the same in the end:

$$\int_C đs = \int_{t=\pi}^{0} 2\,|\mathrm{d}t| = \int_{t=\pi}^{0} 2(-\mathrm{d}t) = -\int_{t=\pi}^{0} 2\,\mathrm{d}t = -(-2\pi) = 2\pi.$$

(But it's simpler to always set things up so that the parameter is increasing.)

Pseudooriented curves

In 2 dimensions, you'll sometimes be asked to integrate a vector field *across* a curve rather than *along* it as usual. Although there is no standard notation for this, you can write it as as $\mathbf{F} \times \mathrm{d}\mathbf{r}$ in analogy with the usual $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$. The book sometimes writes $\mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s$, where $\mathbf{n} = \times\mathbf{T}$ and $\mathrm{d}\mathbf{r} = \mathbf{T}\,\mathrm{d}s$, but this just results in $\mathbf{F} \cdot \times\mathrm{d}\mathbf{r} = \mathbf{F} \times \mathrm{d}\mathbf{r}$. This is the 2-dimensional cross product, so the result is still a scalar. Technically, however, it is actually a **pseudoscalar**, because which scalar it is depends on how you orient the plane (counterclockwise as is the convention or clockwise instead). Similarly, specifying a direction across a curve really gives the curve a **pseudoorientation**, because it only defines a direction along the curve (an orientation) by picking a convention about how these directions correspond. In practice, we orient the plane counterclockwise, meaning that counterclockwise cross products are positive, the rotation $\times\mathbf{v}$ of a vector $\mathbf{v}$ is obtained by rotating it clockwise, a direction across a curve turns into a direction along it by rotation counterclockwise, and a direction along a curve turns into a direction across it by rotating clockwise. But if you consistently did all of these the other way, then the results of all integrals would be the same.

For example, to integrate $\langle 2x, 3y\rangle$ across our semicircle, now pseudooriented upwards, integrate

$$\langle 2x, 3xy\rangle \times \langle \mathrm{d}x, \mathrm{d}y\rangle = 2x\,\mathrm{d}y - 3xy\,\mathrm{d}x,$$

and use the orientation counterclockwise from upwards, which is leftwards (the same as in first example):

$$\begin{aligned}\int_{\substack{x^2+y^2=4,\ y\geq 0\\ \mathrm{d}y\geq 0}} \langle 2x, 3xy\rangle \times \mathrm{d}\mathbf{r} &= \int_{\substack{x^2+y^2=4,\ y\geq 0\\ \mathrm{d}x\leq 0}} (2x\,\mathrm{d}y - 3xy\,\mathrm{d}x)\\ &= \int_{t=0}^{\pi} \Big((2(2\cos t)(2\cos t\,\mathrm{d}t)) - 3(2\cos t)(2\sin t)(-2\sin t\,\mathrm{d}t)\Big)\\ &= \int_{t=0}^{\pi} (8\cos^2 t + 24\sin^2 t\cos t)\,\mathrm{d}t = 4\pi.\end{aligned}$$

Since the vector $\langle 2x, 3xy\rangle$ points to the right where we cross the curve to the right (on the right side) and points to the left where we cross to the left, this suggests that the horizontal component should give a positive result. However, since this vector points upwards on the right side and downwards on the left side, while we cross the curve consistently upwards, this suggests that the vertical component should cancel. So you should again expect a positive result before doing the calculation.