# Exterior differential forms

All of the integrals in vector calculus can be thought of as integrals of *differential forms* of one sort or another. Since integration of differential forms generalizes in ways that integration of vector fields cannot (some of which are important in applications, especially to physics), it's useful to be able to think about differential forms. Furthermore, you then need fewer formulas for the various derivatives of vector fields and for the theorems that relate derivatives to integrals.

## General principles

Here I spell out the general principles of integrating differential forms, but it's really the examples that follow that will make the ideas clear.

There are three sorts of differential forms that we'll need: *exterior* forms, *pseudoexterior* forms, and *absolute* forms. The exterior forms are the most straightforward kind and the simplest to calculate with. The pseudoexterior forms are essentially the same as exterior forms, except that their sign is determined by using the right-hand rule; if you used the left-hand rule instead, then the pseudoexterior forms would have opposite sign but the results of all integrals would stay the same. (In general, you can put 'pseudo' before the name of a concept to get the name of a related concept where the sign depends on the right-hand rule. It is sometimes handy to keep track of whether something is pseudo or not; for example, if you ever add something pseudo to something nonpseudo, then you know that you're making a mistake, much as you would be if you added quantities measured in different units. However, you can ignore the difference in calculations as long as you always use the right-hand rule.) The absolute forms are least used in applications; they typically arise by taking the absolute value of another form (and then possibly multiplying by a scalar quantity). However, they are still important, since lengths, areas, and volumes may be found by integrating absolute forms. (If you read other material on differential forms, the exterior ones are the most commonly studied, and people will often leave out the word 'exterior'. Then the pseudoexterior forms are just called 'pseudoforms', and there is no common name for the absolute forms at all; 'absolute' is a term for them that I made up. On the other hand, there are yet other kinds of differential forms besides all of these.)

You integrate these forms along various regions in space, called *manifolds*. These manifolds can correspondingly be oriented, pseudooriented, or unoriented. Now it's the unoriented manifolds that are the simplest; they are just shapes of consistent dimension. With an oriented manifold, you also make a choice of which direction to go along the manifold; with a pseudooriented manifold, you instead make a choice of which direction to go around or across the manifold. You integrate exterior forms on oriented manifolds, pseudoexterior forms on pseudooriented manifolds, and absolute forms on unoriented manifolds. (If you read other material, the pseudooriented manifolds are sometimes also called 'transversely oriented'.) People also talk about integrating on *chains*: a chain is just a list of manifolds, each with a real number (its *weight*); to integrate a differential form on a chain, you multiply the integral on each manifold by that manifold's weight and then add these products. You'll see some simple examples of chains when we get to the Stokes Theorem below.

To calculate integrals, you want to parametrize your manifolds; you'll have one or more variables $t, u, v, \ldots$ (the *parameters*), running over some domain of values, and a point-valued function (the *parametrization*) of those variables specifying which point in space corresponds to which values of the parameters. Running this function over the entire domain of parameters carves out the manifold. (You'll want your parametrization functions to be continuously differentiable, in order to avoid worrying about whether the integrals are defined. For the same reason, the forms themselves should be continuous, and the domains of the paremetrizations should be compact, that is closed and bounded. The integrals may be defined in other cases, but they are guaranteed to exist if these conditions are met.)

The number of parameters used is the *dimension* of the manifold. This must match the *rank* of the differential form, which is the number of differentials in each term of the form. These differentials are combined using the *wedge product*, $\wedge$. A key property of the wedge product is that it is *anticommutative* between differentials; that is,

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$$

(much like the cross product of vectors). This also means that $\mathrm{d}x \wedge \mathrm{d}x = 0$. However, for absolute forms, you take the absolute value of the wedge product; then $|\mathrm{d}x \wedge \mathrm{d}y| = |-\mathrm{d}y \wedge \mathrm{d}x| = |\mathrm{d}y \wedge \mathrm{d}x|$, while $|\mathrm{d}x \wedge \mathrm{d}x| = |0| = 0$ still.

To calculate the integral, you use the parametrization to express the coordinates $x, y, \ldots$ in terms of the parameters $t, u, v, \ldots$, then differentiate this to get $\mathrm{d}x, \mathrm{d}y, \ldots$ in terms of $\mathrm{d}t, \mathrm{d}u, \mathrm{d}v, \ldots$, so that the integral is entirely in terms of the parameters. You then express this as an iterated integral, checking the orientation or pseudoorientation and putting a minus sign out front if it goes the wrong way.

### Summary of the integrals

This section repeats what we've already done, but shows explicitly how every integral that you deal with in this course is either the integral of an exterior form on an oriented manifold, the integral of a pseudoexterior form on a pseudooriented manifold, or the integral of an absolute form on an unoriented manifold.

*Curves*

A **curve** $C$ is a manifold of dimension 1. So it may be parametrized by a function (which we'll assume is continuously differentiable) that takes one variable $t$ to a point $R = (x, y, \ldots)$. Note that the differential $\mathrm{d}R = \langle \mathrm{d}x, \mathrm{d}y, \ldots \rangle$ is a vector; if you write $\mathbf{r}$ for the vector $R - (0, 0, \ldots)$, then $\mathrm{d}R = \mathrm{d}\mathbf{r}$, and $\mathrm{d}\mathbf{r}$ is the more usual notation (even though $R$ is the more fundamental concept). When you orient a curve, you specify which direction to travel along the curve; when you pseudoorient a curve in 2 dimensions, you specify which direction to travel across the curve. (You won't need to pseudoorient a curve in more dimensions in this class, although it can be done by specifying directions around the curve.)

To integrate a vector quantity $\mathbf{F} = \langle M, N, \ldots \rangle$ along an oriented curve $C$, you integrate the rank-1 exterior form $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \langle M, N, \ldots \rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \ldots \rangle = \int_C (M\,\mathrm{d}x + N\,\mathrm{d}y + \cdots) = \int_C \left( M \frac{\mathrm{d}x}{\mathrm{d}t} + N \frac{\mathrm{d}y}{\mathrm{d}t} + \cdots \right) \mathrm{d}t$$

or

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_C \langle M, N, \ldots \rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots \right\rangle \mathrm{d}t = \int_C \left( M \frac{\mathrm{d}x}{\mathrm{d}t} + N \frac{\mathrm{d}y}{\mathrm{d}t} + \ldots \right) \mathrm{d}t.$$

(There's no need to learn all of these formulas; just put everything in terms of $t$ and push through.) To match orientations, make sure that the direction along the curve as $t$ increases is the same direction as the curve's orientation; or if not, then put a minus sign out front.

To integrate a vector quantity $\mathbf{F} = \langle M, N \rangle$ across a pseudooriented curve $C$ in 2 dimensions, you integrate the rank-1 pseudoexterior form $\mathbf{F} \times \mathrm{d}\mathbf{r}$ (where the cross product in 2 dimensions produces a scalar, or rather a pseudoscalar since the sign depends on the right-hand rule):

$$\int_C \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_C \langle M, N \rangle \times \langle \mathrm{d}x, \mathrm{d}y \rangle = \int_C (M\,\mathrm{d}y - N\,\mathrm{d}x) = \int_C \left( M \frac{\mathrm{d}y}{\mathrm{d}t} - N \frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t$$

or

$$\int_C \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_C \langle M, N \rangle \times \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t} \right\rangle \mathrm{d}t = \int_C \left( M \frac{\mathrm{d}y}{\mathrm{d}t} - N \frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t.$$

To match pseudoorientations using the right-hand rule, make sure that the direction along the curve as $t$ changes is counterclockwise from the direction of the curve's pseudoorientation; or if not, then put a minus sign out front.

To integrate a scalar quantity $f$ on an unoriented curve $C$, you integrate the rank-1 absolute form $f\,\text{đ}s$, where $s$ has no meaning by itself but instead $\text{đ}s$ is the absolute form $\|\mathrm{d}\mathbf{r}\|$:

$$\int_C f\,\text{đ}s = \int_C f\,\|\mathrm{d}\mathbf{r}\| = \int_C f\,\|\langle \mathrm{d}x, \mathrm{d}y, \ldots \rangle\| = \int_C f \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + \cdots} = \int_C f \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \cdots}\,|\mathrm{d}t|$$

or

$$\int_C f\,\text{đ}s = \int_C f\,\|\mathrm{d}\mathbf{r}\| = \int_C f \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| |\mathrm{d}t| = \int_C f \left\| \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots \right\rangle \right\| |\mathrm{d}t| = \int_C f \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \cdots}\,|\mathrm{d}t|.$$

Now there is no orientation to match; instead, make sure that $t$ is increasing, so that $|\mathrm{d}t| = \mathrm{d}t$ in the integral; or if not, then put a minus sign out front.

*Surfaces*

A **surface** $S$ is a manifold of dimension 2, given by a function (which we'll assume is continuously differentiable) that takes two variables $u, v$ to a point $R = (x, y, z, \ldots)$ When you pseudoorient a surface in 3 dimensions, you specify which direction to travel across the surface. (You won't need to pseudoorient a surface in more dimensions, nor will you need to orient any at all, although again these can be done.)

To integrate a vector quantity $\mathbf{F} = \langle M, N, O \rangle$ across a pseudooriented surface $S$ in 3 dimensions, you integrate the rank-2 pseudoexterior form $\mathbf{F} \cdot đ\mathbf{S}$, where $\mathbf{S}$ has no meaning by itself, but instead $đ\mathbf{S}$ is the pseudovector-valued form $1/2\, \mathrm{d}\mathbf{r} \hat{\times} \mathrm{d}\mathbf{r}$ (which as a vector is multiplied by the cross product and as a differential form is multiplied by the wedge product). This works out to $\langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y \rangle$ (using the right-hand rule) or $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v \, \mathrm{d}u \wedge \mathrm{d}v$:

$$\int_S \mathbf{F} \cdot đ\mathbf{S} = \int_S \langle M, N, O \rangle \cdot \langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y \rangle = \int_S (M \, \mathrm{d}y \wedge \mathrm{d}z + N \, \mathrm{d}z \wedge \mathrm{d}x + O \, \mathrm{d}x \wedge \mathrm{d}y)$$
$$= \int_S \left( M \left( \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right) + N \left( \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right) + O \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \right) \mathrm{d}u \wedge \mathrm{d}v$$

or

$$\int_S \mathbf{F} \cdot đ\mathbf{S} = \int_S \langle M, N, O \rangle \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \, \mathrm{d}u \wedge \mathrm{d}v = \int_S \langle M, N, O \rangle \cdot \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \mathrm{d}u \wedge \mathrm{d}v$$
$$= \int_S \left( M \left( \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right) + N \left( \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right) + O \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \right) \mathrm{d}u \wedge \mathrm{d}v.$$

To match pseudoorientations using the right-hand rule, make sure that, as you turn the fingers of your right hand from the direction in which $u$ changes towards the direction in which $v$ changes, your thumb points in the direction of the surface's pseudoorientation; or if not, then put a minus sign out front.

To integrate a scalar quantity $f$ on an unoriented surface $S$, you integrate the rank-2 absolute form $f \, đ\sigma$, where $\sigma$ has no meaning by itself but instead $đ\sigma$ is the absolute form $\|đ\mathbf{S}\|$:

$$\int_S f \, đ\sigma = \int_S f \, \|đ\mathbf{S}\| = \int_S f \, \|\langle \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y \rangle\| = \int_S f \sqrt{(\mathrm{d}y \wedge \mathrm{d}z)^2 + (\mathrm{d}z \wedge \mathrm{d}x)^2 + (\mathrm{d}x \wedge \mathrm{d}y)^2}$$
$$= \int_S f \sqrt{\left( \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right)^2} \, |\mathrm{d}u \wedge \mathrm{d}v|$$

or

$$\int_S f \, đ\sigma = \int_S f \, \|đ\mathbf{S}\| = \int_S f \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| |\mathrm{d}u \wedge \mathrm{d}v| = \int_S f \left\| \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \right\| |\mathrm{d}u \wedge \mathrm{d}v|$$
$$= \int_S f \sqrt{\left( \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right)^2} \, |\mathrm{d}u \wedge \mathrm{d}v|.$$

Again there is no orientation to match; instead, make sure that $u$ and $v$ are both increasing, so that $|\mathrm{d}u \wedge \mathrm{d}v| = \mathrm{d}u \, \mathrm{d}v$ in the integral; or if not, then put a minus sign out front for each one that doesn't.

*Area integrals*

The coordinate plane is both an ambient space of dimension 2 and a manifold of dimension 2 within itself. You can parametrize it simply by the coordinates $x$ and $y$, although there are other ways to parametrize it (such as by polar coordinates).

Instead of $đ\mathbf{S}$, we can look at the pseudoexterior form $1/2\, \mathrm{d}\mathbf{r} \hat{\times} \mathrm{d}\mathbf{r}$, which works out to $\mathrm{d}x \wedge \mathrm{d}y$ (using the right-hand rule). Alternatively, instead of $đ\sigma$, we can look at the absolute form $|\mathrm{d}x \wedge \mathrm{d}y|$. These are actually two equivalent ways to think of the area form $đA$, because there is nothing to do to pseudorient a manifold within itself; it's not possible to go around or across the plane while staying within the plane. In the past, we've thought of $đA$ as an absolute form, which means that you didn't have to worry about orientation or the right-hand rule. But when applying the Stokes Theorem later on, you'll have to think of $đA$ as a pseudoexterior form, because the Stokes Theorem doesn't apply to absolute forms in general.

In any case, to integrate a scalar quantity $f$ on a region in the plane, you integrate the rank-2 form $f \, đA$; make sure that $x$ and $y$ are both increasing, so that $|\mathrm{d}x \wedge \mathrm{d}y| = \mathrm{d}x \, \mathrm{d}y$ in the integral; or if not, then put a minus sign out front for each one that doesn't.

*Volume integrals*

Similarly, ordinary three-dimensional space is both an ambient space of dimension 3 and a manifold of dimension 3 within itself. You can parametrize it by the coordinates $x$, $y$, and $z$, although again there are other ways to parametrize it (such as by cylindrical or spherical coordinates).

Instead of $đA$ we can look at the pseudoexterior form $1/6\,\mathrm{d}\mathbf{r} \mathbin{\hat{\cdot}} \mathrm{d}\mathbf{r} \mathbin{\hat{\times}} \mathrm{d}\mathbf{r}$, which works out to $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ (using the right-hand rule), or the absolute form $|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z|$. Again, these are two equivalent ways to think of the volume form $đV$. In the past, we've thought of $đV$ as an absolute form; but when applying the Stokes Theorem later on, you'll have to think of $đV$ as a pseudoexterior form.

In any case, to integrate a scalar quantity $f$ in a region in space, you integrate the rank-3 form $f\,đV$; make sure that $x$, $y$, and $z$ are all increasing, so that $|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z| = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in the integral; or if not, then put a minus sign out front for each one that doesn't.

## The Stokes Theorem

The (second) Fundamental Theorem of Calculus states that

$$\int_a^b \mathrm{d}u = u|_a^b.$$

This works just as well when there are several independent variables as when there is just one. In this case, you can also write $\mathrm{d}(f(R))$ as $\nabla f(R) \cdot \mathrm{d}\mathbf{r}$ to get the theorem

$$\int_{R=a}^b \nabla f(R) \cdot \mathrm{d}\mathbf{r} = f(b) - f(a).$$

Although this is now a theorem about integrating a gradient along a curve, in essence it is still just the FTC, a theorem about integrating differentials.

To keep the notation simple, I'll continue to refer to scalar- and vector-valued quantities rather than to scalar and vector fields (which are kinds of functions). The only real imprecision here is that the symbol written '$\nabla$' should properly be $\mathrm{d}/\mathrm{d}R$ (or $\mathrm{d}/\mathrm{d}\mathbf{r}$) to indicate the variables with respect to which you're differentiating; however, '$\nabla$' is much more common. So for example, I'll write the preceding statement about gradients as

$$\int_a^b \nabla f \cdot \mathrm{d}\mathbf{r} = f|_a^b,$$

where the $f$ here is really the same as what was $u$ before.

This theorem generalizes to differential forms of higher rank, where it is called the **Stokes Theorem**:

$$\int_M \mathrm{d} \wedge \alpha = \int_{\partial M} \alpha.$$

Here, $\alpha$ is any exterior or pseudoexterior differential form and $M$ is any oriented or pseudooriented manifold, so long as they have the same kind of orientation and the dimension of $M$ is 1 more than the rank of $\alpha$ (so that the dimensions and ranks in each integral match up). To do this properly, you need to know two things: how to take the differential of a differential form, which is the $\mathrm{d} \wedge \alpha$ in the Stokes Theorem; and how to take the endpoints of a manifold other than a curve, which is the $\partial M$ in the Stokes Theorem (which traditionally, but unfortunately, uses the same symbol as for partial derivatives).

With endpoints, you're really dealing with the *boundary* of a manifold. The boundary of a curve oriented from $a$ to $b$ consists of both the point $\{a\}$ and the point $\{b\}$, the former negatively and the latter positively. (Technically, this is a chain: the point $\{a\}$ has weight $-1$, while the point $\{b\}$ has weight 1.) If you think of a point $\{a\}$ as a manifold of dimension 0 and think of a scalar quantity $f$ as a differential form of rank 0, then you integrate $f$ on $\{a\}$ by simply taking the value of $f$ at $a$: $\int_{\{a\}} f = f|_a$, so $\int_{-1\{a\}+1\{b\}} f = -1f|_a + 1f|_b = f|_a^b$. Then the FTC can be written as

$$\int_C \mathrm{d}f = \int_{\partial C} f.$$

The boundary of a surface is a curve (or a chain made up of several curves), and the boundary of a region of space is a surface (or a chain made up of several surfaces).

When you take the differential of an exterior differential form $\alpha$, you get another exterior differential form if you use the *exterior* differential $\mathrm{d} \wedge \alpha$ (which is usually written just '$\mathrm{d}\alpha$' by people who study only exterior and pseudoexterior forms, even though there is also an ordinary nonexterior differential that could be used instead). You can also apply this to a pseudoexterior form to get another pseudoexterior form. When you add forms, the exterior differential obeys the Sum Rule as usual; when you multiply them, you have a kind of Product Rule too. This is the same as the usual Product Rule, except that you must keep track of the order of multiplication. However, this caveat really doesn't matter due to the next rule: the exterior differential of a differential is zero. For example,

$$\mathrm{d} \wedge (x\,\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + x\,\mathrm{d} \wedge \mathrm{d}y \wedge \mathrm{d}z - x\,\mathrm{d}y \wedge \mathrm{d} \wedge \mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + 0 - 0 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

So in the end, you just take the differential of the non-differential factor of each term, then stick this with a wedge in front of the previous differential factors.

When you relate differential forms to vector fields, you can also use various ways of taking derivatives of vector fields. These can be expressed using $\nabla$ and one of the ways of multiplying vectors: the **divergence** $\nabla \cdot \mathbf{F}$ is a scalar field, and the **curl** $\nabla \times \mathbf{F}$ is a pseudovector field in 3 dimensions or a pseudoscalar field in 2 dimensions. Specifically, if $\mathbf{F}(x, y, \ldots) = \langle M, N, \ldots \rangle$, then

$$\nabla \cdot \mathbf{F}(x, y, \ldots) = \langle \partial/\partial x, \partial/\partial y, \ldots \rangle \cdot \langle M, N, \ldots \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \cdots;$$

and

$$\nabla \times \mathbf{F}(x, y, z) = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \times \langle M, N, O \rangle = \left\langle \frac{\partial O}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial O}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

in 3 dimensions, while

$$\nabla \times \mathbf{F}(x, y) = \langle \partial/\partial x, \partial/\partial y \rangle \times \langle M, N \rangle = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

in 2 dimensions.

The connection between these and differentials is as follows (where now I'll conflate the functions $f$ and $\mathbf{F}$ with their values $f(x, y, \ldots)$ and $\mathbf{F}(x, y, \ldots)$ to keep the notation short):

- $\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r}$ in any number of dimensions;
- $\mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla \times \mathbf{F}\,đA$ in 2 dimensions;
- $\mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla \times \mathbf{F} \cdot đ\mathbf{S}$ in 3 dimensions;
- $\mathrm{d} \wedge (\mathbf{F} \times \mathrm{d}\mathbf{r}) = \nabla \cdot \mathbf{F}\,đA$ in 2 dimensions; and
- $\mathrm{d} \wedge (\mathbf{F} \cdot đ\mathbf{S}) = \nabla \cdot \mathbf{F}\,đV$ in 3 dimensions.

(These are not new principles, but rather facts that you can verify by writing everything in terms of the components of $\mathbf{F}$, partial derivatives, and differentials.) Here, $đA$ is the area form $|\mathrm{d}x \wedge \mathrm{d}y|$, which you should now think of as a pseudoexterior form that you can identify with $\mathrm{d}x \wedge \mathrm{d}y$ using the right-hand rule, and $đV$ is the volume form $|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z|$, which you should now think of as a pseudoexterior form that you can identify with $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ using the right-hand rule.

Now suppose that a surface $S$ is bounded by a curve $\partial S$. The Stokes Theorem tells you that

$$\int_S \mathrm{d} \wedge \alpha = \int_{\partial S} \alpha,$$

where $\alpha$ is any (exterior or pseudoexterior) differential form of rank 1. If you integrate a vector quantity $\mathbf{F}$ along $\partial S$, then you're really integrating the differential form $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$, so

$$\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S \mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \int_S \nabla \times \mathbf{F} \cdot đ\mathbf{S}$$

in 3 dimensions, or

$$\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S \mathrm{d} \wedge (\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \int_S \nabla \times \mathbf{F}\,đA$$

in 2 dimensions (where $S$ is now a region in the plane). These are the theorems traditionally called *Stokes's Theorem* and *Green's Theorem*, respectively. If, in 2 dimensions, you integrate $\mathbf{F}$ across $\partial S$, then

$$\int_{\partial S} \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_S \mathrm{d} \wedge (\mathbf{F} \times \mathrm{d}\mathbf{r}) = \int_S \nabla \cdot \mathbf{F} \, \text{đ}A,$$

which is another form of Green's Theorem; in terms of differentials, it's just like the previous version, except that the form being integrated is pseudoexterior instead of exterior. (These theorems are not new principles either, but follow from the general Stokes Theorem and the exterior differentials listed above.)

Next, suppose that a region $Q$ in space is bounded by a surface $\partial Q$. Now the Stokes Theorem tells you that

$$\int_Q \mathrm{d} \wedge \alpha = \int_{\partial Q} \alpha,$$

where now $\alpha$ is any (exterior or pseudoexterior) differential form of rank 2. If you integrate a vector field $\mathbf{F}$ across $\partial Q$, then you're really integrating $\mathbf{F} \cdot \text{đ}\mathbf{S}$, so

$$\int_{\partial Q} \mathbf{F} \cdot \text{đ}\mathbf{S} = \int_Q \mathrm{d} \wedge (\mathbf{F} \cdot \text{đ}\mathbf{S}) = \int_Q \nabla \cdot \mathbf{F} \, \text{đ}V.$$

This is the theorem traditionally called *Gauss's Theorem*, although many textbooks simply call it the *Divergence Theorem*. (Once more, you can verify these by explicit calculation.)

Since the boundary $\partial M$ for any manifold is closed in on itself, the boundary of the boundary, $\partial\partial M$, is always empty. This means that

$$\int_M \mathrm{d} \wedge \mathrm{d} \wedge \alpha = \int_{\partial M} \mathrm{d} \wedge \alpha = \int_{\partial\partial M} \alpha = 0;$$

since this is true no matter how small $M$ may be, you can conclude that

$$\mathrm{d} \wedge \mathrm{d} \wedge \alpha = 0$$

for any (exterior or pseudoexterior) differential form $\alpha$. In terms of vector fields, this has two consequences:

$$\nabla \times \nabla f = 0$$

in 2 or 3 dimensions, and

$$\nabla \cdot \nabla \times \mathbf{F} = 0$$

in 3 dimensions. If you write these facts out using partial derivatives, then you'll see that they simply state the equality of mixed partial derivatives. (As a technicality, that equality is not always guaranteed, but it is guaranteed when the mixed partial derivatives are continuous; we derived these facts by considering integrals that likewise are only guaranteed to exist when the forms being integrated are continuous. Conversely, the Stokes Theorem can be proved in the first place by using the equality of mixed partial derivatives and the ordinary FTC applied to iterated integrals, by carefully keeping track of everything.)

**Optional material**

This material doesn't come up in the course, but it's used a lot and fills in some gaps in the concepts.

*Hodge duals*

You may notice that a vector quantity $\mathbf{F}$ can be turned into a differential form in two different ways: in 2 dimensions, $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is an exterior form of rank 1, while $\mathbf{F} \times \mathrm{d}\mathbf{r}$ is a pseudoexterior form of rank 1; in 3 dimensions, $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is again an exterior form of rank 1, while now $\mathbf{F} \cdot đ\mathbf{S}$ is a pseudoexterior form of rank 2. Either way, the two differential forms related to a single vector field are called *Hodge duals* of each other. If you work directly with differential forms instead of vectors, then you can use the Hodge duals to bring in geometric ideas of length and angle. In this way, you can work as much as possible with the objects that you integrate to get measurable quantities.

The Hodge dual of a differential form $\alpha$ is denoted $*\alpha$. In rectangular coordinates, it's easy to calculate Hodge duals; you replace the differential factors of each term with whatever is missing in the area or volume form (written in the order given by the right-hand rule), paying attention to the sign. This gives you

$$*\mathrm{d}x = \mathrm{d}y,\ *\mathrm{d}y = -\mathrm{d}x$$

in 2 dimensions; and

$$*\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z,\ *\mathrm{d}y = -\mathrm{d}x \wedge \mathrm{d}z = \mathrm{d}z \wedge \mathrm{d}x,\ *\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y$$

and

$$*(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x,\ *(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}y,\ *(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z$$

in 3 dimensions. (The Hodge dual of an exterior form is a pseudoexterior form and vice versa, and these rules are written using the right-hand rule.) Now you can check that

$$*(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \mathbf{F} \times \mathrm{d}\mathbf{r},\ *(\mathbf{F} \times \mathrm{d}\mathbf{r}) = -\mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

in 2 dimensions; and

$$*(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \mathbf{F} \cdot đ\mathbf{S},\ *(\mathbf{F} \cdot đ\mathbf{S}) = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

in 3 dimensions. You can even extend this to forms of top rank and to scalar quantities (which are differential forms of rank 0):

$$*(đA) = *(\mathrm{d}x \wedge \mathrm{d}y) = 1,\ *1 = \mathrm{d}x \wedge \mathrm{d}y = đA$$

in 2 dimensions; and

$$*(đV) = *(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = 1,\ *1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = đV$$

in 3 dimensions.

*Laplacians*

The **Laplacian** of a form $\alpha$ is

$$\Delta\alpha = *\Big(\mathrm{d} \wedge *(\mathrm{d} \wedge \alpha)\Big) \pm \mathrm{d} \wedge *(\mathrm{d} \wedge *\alpha),$$

where you use $+$ or $-$ on the second term depending on whether the ambient space has even or odd dimension, and you must throw in another overall minus sign if both the space's dimension and the form's rank are odd. In other words, take the exterior differential, then the Hodge dual, then repeat; and also do this in reverse order; then add or subtract these according to the parity of the dimension, and possibly take the opposite of the entire result. (I know, that's kind of a complicated rule; it's been chosen just so to make everything below work out.) Notice that $\Delta\alpha$ has both the same rank and the same orientation as $\alpha$, so it is a nice notion of second derivative.

If you think of a scalar field $f$ as an exterior form of rank 0, then $\mathrm{d} \wedge f = \mathrm{d}f$, while $*f$ has top rank, so $\mathrm{d} \wedge *f = 0$. Then

$$\Delta f = *(\mathrm{d} \wedge *\mathrm{d}f) = *\Big(\mathrm{d} \wedge *(\nabla f \cdot \mathrm{d}\mathbf{r})\Big) = *\Big(\mathrm{d} \wedge (\nabla f \times \mathrm{d}\mathbf{r})\Big) = *(\nabla \cdot \nabla f\, đA) = \nabla \cdot \nabla f$$

in 2 dimensions; and

$$\Delta f = *(\mathrm{d} \wedge *\mathrm{d}f) = *\Big(\mathrm{d} \wedge *(\nabla f \cdot \mathrm{d}\mathbf{r})\Big) = *\Big(\mathrm{d} \wedge (\nabla f \cdot \text{đ}\mathbf{S})\Big) = *(\nabla \cdot \nabla f \, \text{đ}V) = \nabla \cdot \nabla f$$

in 3 dimensions. In fact, the rule that $\Delta f = \nabla \cdot \nabla f$ is correct in *any* number of dimensions (and the weird rules about minus signs are designed to make that work out); for this reason, the Laplacian operator $\Delta$ is often written as '$\|\nabla\|^2$' or just '$\nabla^2$' (think of $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$).

Other Laplacians are

$$\Delta(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla(\nabla \cdot \mathbf{F}) \cdot \mathrm{d}\mathbf{r} + \nabla(\nabla \times \mathbf{F}) \times \mathrm{d}\mathbf{r},\ \Delta(\mathbf{F} \times \mathrm{d}\mathbf{r}) = \nabla(\nabla \cdot \mathbf{F}) \times \mathrm{d}\mathbf{r} - \nabla(\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{r}$$

in 2 dimensions; and

$$\Delta(\mathbf{F} \cdot \mathrm{d}\mathbf{r}) = \nabla(\nabla \cdot \mathbf{F}) \cdot \mathrm{d}\mathbf{r} - \nabla \times (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{r},\ \Delta(\mathbf{F} \cdot \text{đ}\mathbf{S}) = \nabla(\nabla \cdot \mathbf{F}) \cdot \text{đ}\mathbf{S} - \nabla \times (\nabla \times \mathbf{F}) \cdot \text{đ}\mathbf{S}$$

in 3 dimensions. If you define $\Delta\mathbf{F}$ so that $\Delta\mathbf{F} \cdot \mathrm{d}\mathbf{r} = \Delta(\mathbf{F} \cdot \mathrm{d}\mathbf{r})$, you can see (by working out their components) that $\Delta\mathbf{F} \times \mathrm{d}\mathbf{r} = \Delta(\mathbf{F} \times \mathrm{d}\mathbf{r})$ in 2 dimensions and that $\Delta\mathbf{F} \cdot \text{đ}\mathbf{S} = \Delta(\mathbf{F} \cdot \text{đ}\mathbf{S})$ in 3 dimensions; furthermore, each component of $\Delta\mathbf{F}$ is the Laplacian of the corresponding component of $\mathbf{F}$. So Laplacians work very nicely indeed.