Change of variables in multiple integrals

I often say that the differentials in expressions such as $\int_{x=0}^{1} 3x^2 \,\mathrm{d}x$, $3\,\mathrm{d}x + x^2\,\mathrm{d}y + \mathrm{e}^y\,\mathrm{d}z$, and $\mathrm{d}y/\mathrm{d}x$ can and should be treated literally, not merely as mnemonics for appreciable changes in a limit or an approximation. In particular, you can do a change of variables in a one-dimensional integral by calculating with differentials; for example, to integrate $\sqrt[3]{1-3x}\,\mathrm{d}x$ (say from $x=0$ to $x=3$), let $u = 1-3x$, so that $\mathrm{d}u = -3\,\mathrm{d}x$ (and consequently $\mathrm{d}x = -1/3\,\mathrm{d}u$), and calculate:

$$\int_{x=0}^{3} \sqrt[3]{1-3x}\,\mathrm{d}x = \int_{u=1-3(0)}^{1-3(3)} \sqrt[3]{u}\left(-\frac{1}{3}\,\mathrm{d}u\right) = -\frac{1}{3}\int_{u=1}^{-8} u^{1/3}\,\mathrm{d}u = -\frac{1}{3}\left(\frac{3}{4}u^{4/3}\right)\Big|_{u=1}^{-8} = -\frac{15}{4}.$$

You can even develop a general formula for this change of variables, by solving $u = 1-3x$ for x to get $x = 1/3 - 1/3\,u$ and differentiating that to get $\mathrm{d}x = -1/3\,\mathrm{d}u$ (as before). Then

$$\int_{x=a}^{b} f(x)\,\mathrm{d}x = \int_{u=1-3a}^{1-3b} f\left(\frac{1}{3}-\frac{1}{3}u\right)\left(-\frac{1}{3}\,\mathrm{d}u\right) = -\frac{1}{3}\int_{u=1-3a}^{1-3b} f\left(\frac{1}{3}-\frac{1}{3}u\right)\mathrm{d}u = \frac{1}{3}\int_{u=1-3b}^{1-3a} f\left(\frac{1}{3}-\frac{1}{3}u\right)\mathrm{d}u.$$

(The last step, where I cancelled a minus sign by reversing the order of integration, is not really necessary, but there is a point to it that will become clear towards the end.) Of course, this isn't a formula that's likely to be useful very often, simply because this substitution isn't needed very often, but it does work; for example, using $a=0$, $b=3$, and $f(x) = \sqrt[3]{1-3x}$, you get

$$\int_{x=0}^{3} \sqrt[3]{1-3x}\,\mathrm{d}x = \frac{1}{3}\int_{u=1-3(3)}^{1-3(0)} \sqrt[3]{1-3\left(\frac{1}{3}-\frac{1}{3}u\right)}\,\mathrm{d}u = \frac{1}{3}\int_{u=-8}^{1} \sqrt[3]{u}\,\mathrm{d}u = \frac{1}{3}\left(\frac{3}{4}u^{4/3}\right)\Big|_{u=-8}^{1} = -\frac{15}{4}.$$

This is really the same calculation as before, only with some extra redundant algebra. This same approach also works for integrating along curves as in §12.3 and §§15.1&15.2 of the textbook.

However, this method doesn't seem to work with area and volume integrals. In the case of polar coordinates, §14.4 of the textbook says to use a general formula similar to the one above:

$$\int_{(x,y)\in R} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{(r,\theta)\in G} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

(see the bottom of page 782), where

$$G = \{r,\theta \mid (r\cos\theta, r\sin\theta)\in R,\ r\geq 0,\ 0\leq\theta\leq 2\pi\}.$$

(This G is somewhat complicated to write down, but it just refers to same region as R, only in different coordinates. Switching from $\int_{(x,y)\in R}$ to $\int_{(r,\theta)\in G}$ is the analogue of switching from $\int_{x=a}^{b}$ to $\int_{u=1-3b}^{1-3a}$.) If you try to derive this general formula in the same way as you can derive the general formula for the substitution $u = 1-3x$, then that explains everything except for the $r\,\mathrm{d}r\,\mathrm{d}\theta$ at the end. Starting from $x = r\cos\theta$ and $y = r\sin\theta$, you get $\mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta$ and $\mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta$, so you'd think that

$$\begin{aligned}\mathrm{d}x\,\mathrm{d}y &= (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta)(\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta)\\ &= \cos\theta\sin\theta\,\mathrm{d}r^2 + r\cos^2\theta\,\mathrm{d}r\,\mathrm{d}\theta - r\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}r - r^2\sin\theta\cos\theta\,\mathrm{d}\theta^2\\ &= \sin\theta\cos\theta\,\mathrm{d}r^2 + r\cos(2\theta)\,\mathrm{d}r\,\mathrm{d}\theta - r^2\sin\theta\cos\theta\,\mathrm{d}\theta^2.\end{aligned}$$

But the answer *should* be simply $r\,\mathrm{d}r\,\mathrm{d}\theta$, so what's going on?

The main part of the solution is to realize that the way in which $\mathrm{d}x$ and $\mathrm{d}y$ are multiplied to get the area element $\mathrm{d}x\,\mathrm{d}y$ is *not* ordinary multiplication. It is another kind of multiplication known as **exterior multiplication** but more often called the **wedge product**, because the symbol for it is a wedge '$\wedge$'. The wedge product of differentials is similar to the cross product of vectors in that it is *anticommutative*: $\mathrm{d}u\,\wedge$

$\mathrm{d}v = -\mathrm{d}v \wedge \mathrm{d}u$. A consequence of this is that $\mathrm{d}u \wedge \mathrm{d}u = 0$ (since it must equal its own opposite). Using this, the calculation becomes

$$\begin{aligned}\mathrm{d}x \wedge \mathrm{d}y &= (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta) \wedge (\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta)\\ &= \cos\theta\sin\theta\,\mathrm{d}r \wedge \mathrm{d}r + r\cos^2\theta\,\mathrm{d}r \wedge \mathrm{d}\theta - r\sin^2\theta\,\mathrm{d}\theta \wedge \mathrm{d}r - r^2\sin\theta\cos\theta\,\mathrm{d}\theta \wedge \mathrm{d}\theta\\ &= 0 + r\cos^2\theta\,\mathrm{d}r \wedge \mathrm{d}\theta + r\sin^2\theta\,\mathrm{d}r \wedge \mathrm{d}\theta - 0 = r\,\mathrm{d}r \wedge \mathrm{d}\theta.\end{aligned}$$

It works!

There is another issue that I need to deal with. In changing variables from x to $u = 1 - 3x$, you could change the beginning and ending bounds of integration separately, so that $x = 0$ in the example becomes $u = 1 - 3(0) = 1$ while $x = 3$ becomes $u = 1 - 3(3) = -8$. In double and triple integrals, however, the region of integration could be any compact (closed and bounded) region, and this can't be described by just a few numbers. In particular, there is no distinction, in the boundary of such a region, between a starting point and an ending point. A complete analogue of the shift from $(x, y) \in R$ to $(r, \theta) \in G$, where G is as decribed above, would shift from $x \in [a, b]$ to $u \in I$, where $I = \{u \mid 1/3 - 1/3\,u \in [a, b]\}$. Because $1/3 - 1/3a > 1/3 - 1/3b$ (when $a < b$), $I = [1/3 - 1/3b, 1/3 - 1/3a]$, suggesting an integral $\int_{u=1/3-1/3b}^{1/3-1/3a}$ rather than $\int_{u=1/3-1/3a}^{1/3-1/3b}$, as you get directly from the substitution. If you want a single expression that comes out to $-1/3\int_{u=1-3a}^{1-3b} f(1/3 - 1/3\,u)\,\mathrm{d}u$ or $1/3\int_{u=1-3b}^{1-3a} f\left(\frac{1}{3} - \frac{1}{3}\,u\right)\mathrm{d}u$ but refers to the limits of integration only via $u \in I$, then you can write

$$\int_{u \in I} f\left(\frac{1}{3} - \frac{1}{3}u\right)\,|\mathrm{d}u|.$$

Then if you interpret I as running from $u = 1 - 3a$ to $u = 1 - 3b$, then u is decreasing, so $\mathrm{d}u$ is negative, so $|\mathrm{d}u| = -\mathrm{d}u$, restoring the minus sign; but if you interpret I as running from $u = 1 - 3b$ to $u = 1 - 3a$, then u is increasing, so $\mathrm{d}u$ is positive, so $|\mathrm{d}u| = \mathrm{d}u$, giving no minus sign.

In one dimension, we don't usually do things this way, but in multiple integrals, this is how they're usually set up. So the area is element is really

$$đA = |\mathrm{d}x \wedge \mathrm{d}y|,$$

and the volume element is really

$$đV = |\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z|.$$

(The symbol 'đ' may be used when something is traditionally written with 'd' but is not really a differential, which is the case with $\mathrm{d}A$ and $\mathrm{d}V$.)

With polar coordinates, the absolute value here make no difference. Since $\mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$, you get

$$đA = |\mathrm{d}x \wedge \mathrm{d}y| = |r\,\mathrm{d}r \wedge \mathrm{d}\theta| = r\;|\mathrm{d}r \wedge \mathrm{d}\theta|,$$

because we always use polar coordinates so that $r \geq 0$. In cylindrical coordinates,

$$đV = |\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z| = |r\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z| = r\;|\mathrm{d}z \wedge \mathrm{d}r \wedge \mathrm{d}\theta|.$$

Then in spherical coordinates, since $\mathrm{d}z \wedge \mathrm{d}r = \rho\,\mathrm{d}\rho \wedge \mathrm{d}\phi$ in exactly the same way that $\mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$,

$$đV = |\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z| = |r\,\mathrm{d}z \wedge \mathrm{d}r \wedge \mathrm{d}\theta| = |r\rho\,\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta| = |\rho^2\sin\phi\,\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta| = \rho^2\sin\phi\;|\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta|,$$

because we always use $0 \leq \phi \leq \pi$ so that $\sin\phi \geq 0$. But if you change coordiantes in some other way, then the absolute value may matter. This is why the general formula for change of coordinates in Section 14.8 of the textbook involves an absolute value. (It also involves a determinant, and this takes care of the calculations involving only the wedge product.) But you don't need that formula at all, so long as you know that $đA = |\mathrm{d}x \wedge \mathrm{d}y|$ and $đV = |\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z|$ and are willing to calculate with differentials.