Definitions for functions of several variables

In order to form precise definitions of various concepts related to functions of several variables, it's handy to piggyback on the definitions for functions of one variable. This is *not* the way that the book writes its definitions, but it's the way that I prefer. So here are my definitions.

General principles

Recall that a **parametrized curve**, or **point-valued function**, takes a number to a point (in however many dimensions we're dealing with, typically 2 or 3 dimensions). That is, if C is a parametrized curve and a is a real number, then $C(a)$ is a point $P = (x, y)$ or $P = (x, y, z)$. (assuming that we're dealing with 2 or 3 dimensions). Meanwhile, a function of several variables (however many variables we're dealing with, typically 2 or 3 variables) takes a point to a number; that is, if f is a function of (say) 2 or 3 variables and $P = (x, y)$ or $P = (x, y, z)$ is a point in 2 or 3 dimensions, then $f(P) = f(x, y)$ or $f(x, y, z)$ is a real number c. If we combine these by composition, then $f \circ C$ is an ordinary function; that is, if a is a real number, then so is $(f \circ C)(a)$:

$$(f \circ C)(a) = f\big(C(a)\big) = f(P) = c.$$

From one-variable Calculus, you should know how to define various concepts (continuity, limits, differentiability, derivatives, differentials) for ordinary functions. It's easy to extend these concepts to vector- and point-valued functions (parametrized curves), since these simply consist of several ordinary functions (the coordinates or components). So to define these concepts for functions of several variables, we typically use a formula like this:

> If $f \circ C$ has a certain property whenever C does, no matter what C might be (as long as it has the property), then that's what it means for f to have that property.

This formula doesn't always work perfectly; for one thing, we often want to say more than just a Yes/No property, and it may not be obvious what matters about C or how to extract the appropriate information from the composites. Besides that, even when this formula would make perfect sense, sometimes some of the nice theorems that we would expect aren't always true, which means that we should look for a modifed definition that makes the theorems work. (That's what mathematicians really want from a definition: they're not handed down from on high but developed for the purpose of getting correct results.) Nevertheless, all of the definitions here will be based on something like this formula.

Continuity

Continuity follows the general formula precisely. A function f of several variables is **continuous** if, whenever C is a continuous parametrized curve, the composite $f \circ C$ is a continuous function. (It wouldn't be fair to expect $f \circ C$ to be continuous unless C is continuous as well as f, but if both C and f are continuous, then their composite ought to be as well.)

Sometimes we want to look at continuity in more detail; in general, to say that a function is continuous really means that it's continuous at every number in its domain. So for a function of several variables, we want to talk about continuity at particular points in its domain. A function f is **continuous** at a point P in the domain of f if, whenever C is a parametrized curve and a is a number such that $C(a) = P$ and C is continuous at a, then $f \circ C$ is also continuous at a. Again, it wouldn't be fair to demand more than this if we're only asking f to be continuous at P.

An equivalent definition is to say that f is continuous at P if f is deifned at P and, for every positive number ϵ, there is some positive number δ such that, whenever $|Q - P| < \delta$ and f is defined at Q, then $|f(Q) - f(P)| < \epsilon$. However, this is rather less fun to work with. Ultimately, you have to say something like this some time, but I prefer to say it once, when giving the first definition in one-variable Calculus, and then never again.

Any function with a formula that is built out of the coordinate variables using only the usual operations is continuous wherever it is defined. (To be definite, the usual operations are addition, opposites, subtraction, multiplication, reciprocals, division, absolute values, powers with constant exponents, powers with positive bases, roots with constant indexes, roots with constant radicands, logarithms, the six trigonometric operations, and the six inverse trigonometric operations. Some exceptions to this include piecewise definitions and powers where the exponent varies and the base may be zero or negative.) To prove this, you use the continuity of each component of a continuous parameterized curve and the one-variable theorem that any function built out of continuous functions using these operations is continuous.

Limits

To keep things simple, I'll only look at finite limits approaching a finite value; none of our limits will involve infinity in any role. (I'll make things more complicated in another way shortly!)

There is a technicality about limits that's often ignored in one-variable calculus, which is that the expression whose limit you're taking must be defined at numbers arbitrarily close to the number that the variable is approaching. It's often treated as a big deal that the function doesn't have to be defined at that number precisely, which is certainly true and important, but it still has to be defined *near* that number. For example (and assuming that we're only working with real numbers), you can't talk about the limit of $\sqrt{x}$ as $x \to -1$, because x can't get very close to -1 while $\sqrt{x}$ is defined. On the other hand, it's fine to talk about the limit as $x \to 0$, because even though $\sqrt{x}$ is undefined when $x < 0$, still $\sqrt{x}$ is defined when $x > 0$, which allows x to get arbitrarily close to 0. (But on the other other hand, you can't talk about the limit as $x \to 0^-$, because now this requires $x < 0$, which leaves $\sqrt{x}$ undefined again.)

A number a is a **limit point** of a set D if it makes sense to talk about a function defined on D as having a limit approaching a, in other words if there exists a function whose domain is D (a constant function will do) that has a limit approaching a. (The term 'limit point' is traditional even in one dimension, even though we normally call a a number rather than a point.) This is equivalent to saying that there are numbers in D (other than possibly a itself) that are arbitrarily close to a, in other words if, given any positive distance $\delta > 0$, there is at least some number b in the set D such that $0 < |b - a| < \delta$. But I prefer to think of the definition that has no δ (or ϵ) in it.

Keeping this technicality in mind, the **limit** approaching a point P of a function f of several variables (which in symbols we can write as

$$\lim_{(x,y)\to P} f(x, y)$$

or

$$\lim_{(x,y,z)\to P} f(x, y, z)$$

in 2 or 3 dimensions) is the unique number L (if this exists) such that, whenever C is a parametrized curve and a is a number, if $C(t) = P$ when and only when $t = a$, and if C is continuous at a, and if a is a limit point of $f \circ C$, then L is the limit of $f \circ C$ approaching a. In other words (ignoring the fine print), whenever

$$\lim_{t\to a} C(t) = P,$$

then

$$\lim_{t\to a} f\big(C(t)\big) = L.$$

The limit of one of these composites is basically the limit of the function along a particular curve. If the function is undefined along the curve, then we don't expect its limit to exist, and this is what the clause about limit points takes care of. We also don't want to worry about $f(P)$, since f might not be continuous, which is why $C(a)$ is not allowed to be P except when $t = a$. Then, in order for the limit to exist overall, the limit must exist along each appropriate curve and be the same along all of them.

If for any appropriate curve, there is no limit along that curve, then the limit overall does not exist. Besides that, if there are two such curves such that the limits along them are different, then again the limit does not exist overall. It is in this way that one generally proves that a limit does not exist, when it doesn't . When the limits does exist, however, then you usually need to find a general argument to show that it does and what it is, because you can't actually check every individual curve.

One often talks about limits with restrictions on the manner of approaching the point. For example, instead of saying (x, y) approaches $(2, 3)$, we might say that (x, y) approaches $(2, 3)$ *while* $x \neq y$. Technically, this is handled by modifying the function so that it is undefined when $x = y$ (in this case).

Differentiability

The way that differentiability fits in with composition of functions is the chain rule $(f \circ g)'(x) = f'\big(g(x)\big)g'(x)$. Following the general principle, we replace g with a parametrized curve C, and the values of the derivatives of this (replacing $g'(x)$) are vectors. However, the composite is an ordinary function, so the derivative of f should multiply by a vector to get a scalar. One way to do this is to multiply a vector by a vector with the dot product, so the derivative of a function of several variables should be a vector. There are actually several sorts of derivatives in higher dimensions, and I'll come back to this subject later; but the one which is a vector will provide the definition of differentiability.

We say that the function f is **differentiable** at some point P if there exists a vector $\mathbf{v}$ such that, whenever C is a parametrized curve and a is a number such that $C(a) = P$ and C is differentiable at a, then $f \circ C$ is also differentiable at a and furthermore $(f \circ C)'(a) = \mathbf{v} \cdot C'(a)$. If f is differentiable at every point P in its domain, then f is simply *differentiable*.

This vector $\mathbf{v}$ is called the **gradient** of f at P and may be written as $\nabla f(P)$ (although $f'(P)$ would make a lot of sense), so the rule is

$$(f \circ C)'(a) = \nabla f(P) \cdot C'(a).$$

Higher differentiability

If a function f is differentiable, or more generally where it is differentiable, the components of its gradients define some more functions, called the **partial derivatives** of f. (We will look at these partial derivatives more later on.) Wherever the partial derivatives are themselves continuous, the original function is **continuously differentiable**. Where the partial derivatives are themselves differentiable, the original function is **twice differentiable**. Where the partial derivatives are continuously differentiable, the original function is **twice continuously differentiable**. Etc etc etc.

Where this goes on forever, the function is **infinitely differentiable**. Any function built out of the usual operations is infinitely differentiable except at certain exceptional places where a derivative fails to exist, such as when taking the absolute value or square root of zero. But to prove this, it's best to look at how to calculate the derivatives, which I'll come back to later on.