Fubini theorems

I want to record here some theorems about double (and higher) Riemann integrals, culminating in using the Fubini theorems to turn them into iterated integrals.

First, I want to note some notation that is a little more precise than the notation in the textbook. The notation in the textbook is very common, and it's usually quite clear what it means, but it's not good enough if you want to be completely unambiguous about what variables you're using and where. So rather than write, for example,

$$\iint_D f(x,y)\,\mathrm{d}A,$$

where D is a region in 2 dimensions (formally a relation between 2 variables) and f is a function of 2 variables, I'll write

$$\int_{(x,y)\in D} f(x,y)\,|\mathrm{d}x \wedge \mathrm{d}y|.$$

So to begin with, since the integrand (all of the stuff after the integral symbols) makes it clear that there are 2 variables of integration, it's not necessary to repeat the integral symbol. But just as we write $f(x,y)$ (rather than just f) after that symbol to indicate the value of the function f at particular values of the variables x and y (rather than its value somewhere else), so I write $(x,y) \in D$ (rather than just D) beneath that symbol to indicate that the point whose coordinates are those values (rather than some other point) belongs to the region D.

At the end, since $\mathrm{d}A$ (and $\mathrm{d}V$ in 3 dimensions) don't indicate which variables are being used, I use the notation $|\mathrm{d}x \wedge \mathrm{d}y|$ (or $|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z|$ in 3 dimensions). This notation is more complicated than necessary just to indicate the variables, but there is a reason for it; just as the notation $\mathrm{d}y/\mathrm{d}x$ for a derivative is not merely an arbitrary symbol but can be literally understood as the result of dividing expressions (called differentials) obtained by applying an operator d, so the notation $|\mathrm{d}x \wedge \mathrm{d}y|$ for an area element (or $|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z|$ for a volume element) is not merely an arbitrary symbol but can be literally understood as the absolute value of an expression (called an exterior differential form) involving an operator $\wedge$. However, don't worry about that for now; just treat it as a notation used to indicate precisely which variables are used in the area (or volume) element.

Using that notation, here are the important theorems:

1 The integral of a continuous function on a compact (that is closed and bounded) region always exists: $\int_{(x,y)\in D} f(x,y)\,|\mathrm{d}x \wedge \mathrm{d}y|$ exists if f is continuous and D is compact (and similarly in more variables).

2 If two regions D_1 and D_2 are completely disjoint (no overlap at all), or if their overlap is contained within a single point/line/plane/etc of fewer dimensions than the overall number of variables, and if a function f has integrals on both of these regions, then the integral of f on their union (the combined region $D_1 \cup D_2$) also exists and is the sum of the separate integrals:

$$\int_{(x,y)\in D_1\cup D_2} f(x,y)\,|\mathrm{d}x \wedge \mathrm{d}y| = \int_{(x,y)\in D_1} f(x,y)\,|\mathrm{d}x \wedge \mathrm{d}y| + \int_{(x,y)\in D_2} f(x,y)\,|\mathrm{d}x \wedge \mathrm{d}y|$$

(and similarly in more variables) if the integrals on the right exist and the overlap is small.

3 In any double (or higher) integral, if two of the variables are swapped in both the function being integrated and in the region over which it is integrated (or equivalently, by renaming the variables, by swapping the variables only with the area/volume/etc element), then the result is the same (so that if either integral exists, then so does the other, and then they are equal):

$$\int_{(x,y)\in D} f(x,y)\,|\mathrm{d}x \wedge \mathrm{d}y| = \int_{(x,y)\in D} f(x,y)\,|\mathrm{d}y \wedge \mathrm{d}x|$$

(and similarly in more variables).

4 For a region D in 2 dimensions, if there are constants a and b with $a \leq b$ and continuous functions g and h (each of 1 variable) such that $(x, y) \in D$ if and only if $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$, and if $g(x) \leq h(x)$ whenever $a \leq x \leq b$, then the integral of any continuous function f on D is the same as the corresponding iterated integral:

$$\int_{(x,y)\in D} f(x, y)\, |\mathrm{d}x \wedge \mathrm{d}y| = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} \mathrm{d}y \right) \mathrm{d}x.$$

5 For a region D in 3 (or more) variables, if there are a compact region R in 2 variables (or in general a compact region of one fewer dimension) and continuous functions g and h of 2 variables each (or in general with the same number of variables as R has dimensions) such that $(x, y, z) \in D$ if and only if $(x, y) \in R$ and $g(x, y) \leq z \leq h(x, y)$, and if $g(x, y) \leq h(x, y)$ whenever $(x, y) \in R$, then the integral of any continuous function f on D is the same as the corresponding iterated integral:

$$\int_{(x,y,z)\in D} f(x, y, z)\, |\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z| = \int_{(x,y)\in R} \left(\int_{z=g(x,y)}^{h(x,y)} f(x, y, z)\, \mathrm{d}z \right) |\mathrm{d}x \wedge \mathrm{d}y|$$

(and similarly in more variables).

The last two of these are the Fubini Theorem (for Riemann integrals of continuous functions).

By itself, the Fubini Theorem only works for regions of particular shapes, but the other theorems combine to make it more useful. First of all, Theorem 3 allows us to put the variables in whatever order we like. Even so, the regions still require particular shapes; we can just orient those however we wish. Theorem 2, in principle, allows us to divide a region up into smaller regions appropriate for the Fubini Theorem; the only question is whether the integrals exist. Theorem 1 guarantees this existence for continuous functions.

So using these in order, if you want to integrate over a crazy region, then divide the region into pieces of suitable shape. If the function is continuous and these smaller regions are all compact, then you know that their integrals exist; and if the regions overlap only slightly, then you can recover the answer to the original problem by adding them up. Finally, to get the integrals on these small regions, think of the variables as coming in whichever order works best, and use the Fubini Theorem (possibly more than once) to replace double and triple integrals with iterated integrals. Hopefully, these will be integrals that you can do!