Point- and vector-valued functions

Besides individual points and vectors, one can also consider variable points and vectors, which are the outputs of point- and vector-valued functions. A **point-valued function** in $\mathbf{R}^n$ consists of n ordinary functions, all with the same domain. For example, a point-valued function in $\mathbf{R}^2$ consists of 2 functions with the same domain, say $f(t) = t^2$ and $g(t) = t^3$. We put these together into a single function (f, g), which takes a real-number t as input and produces the point

$$(f, g)(t) = \big(f(t), g(t)\big) = (t^2, t^3) = \mathrm{O} + t^2\mathbf{i} + t^3\mathbf{j}$$

(in this case) as output. A **vector-valued function** in $\mathbf{R}^n$ also consists of n ordinary functions, all with the same domain. But now we think of the output as a vector:

$$\langle f, g\rangle(t) = \langle f(t), g(t)\rangle = \langle t^2, t^3\rangle = t^2\mathbf{i} + t^3\mathbf{j}$$

(for example). If we want to know whether one of these functions is continuous or differentiable, then we just look at each of its components separately. For example, since the functions f and g above are continuous and differentiable everywhere, so are (f, g) and $\langle f, g\rangle$.

The textbook often doesn't distinguish between a point P and its position vector $\mathbf{r} = P - \mathrm{O}$, where O is the origin of a coordinate system. Conceptually, they're very different, since you can talk about points and vectors geometrically without bringing coordinates into it, so the concepts are meaningful even if there is no inherent point O to equivocate them. On the other hand, when doing calculations, it's easy to conflate them; since the coordinates of O are all 0, when you do the subtraction, the components of $\mathbf{r}$ are exactly the same as the coordinates of P. Still, you should always keep in mind whether a given expression really refers to a point or to a vector. In particular, a point-valued function can be viewed as a **parametrized curve**; each value of the input t (which in this context is called a *parameter*) gives a point, and all of these points together make up a curve. A vector-valued function only defines a curve by interpreting each vector with reference to point O deemed to be the origin, but that is how the textbook insists on doing it starting in Chapter 12.

If P is a point, then the difference ΔP is a vector (because it's the result of subtracting two points), and then the differential $\mathrm{d}P$ is an infinitesimal vector. If P is a function of some scalar quantity t, then $\mathrm{d}P/\mathrm{d}t$ makes sense, because it's a vector divided by a scalar, but now it's no longer infinitesimal (unless it happens to be zero). In other words, *the derivative of a point with respect to a scalar is a vector.* Another way to see this is that if F is a point-valued function, then its derivative F' is a vector-valued function:

$$F'(t) = \lim_{h \to 0} \left(\frac{F(t+h) - F(t)}{h}\right);$$

first subtract two points to get a vector, then divide by the scalar h to get another vector, and finally take the limit of these vectors to get a vector. Of course, the derivative of a *vector* with respect to a scalar is *also* a vector; in other words, the derivative of a vector-valued function is also a vector-valued function.

For example, if P gives the position of some object at time t, then P is a point, but $\mathrm{d}P/\mathrm{d}t$, the *velocity* of the object, is a vector. (Note that the magnitude of this vector is the object's *speed*.) If we write $\mathbf{v}$ for $\mathrm{d}P/\mathrm{d}t$ (which can also be written as $\mathrm{d}\mathbf{r}/\mathrm{d}t$), then $\mathrm{d}\mathbf{v}/\mathrm{d}t$ is the acceleration of the object, which is also a vector. (Physicists and mechanical engineers use the word 'acceleration' like this, to indicate any change in velocity —speed or direction— over time. In everyday language, this word means something more like $\mathrm{d}\|\mathbf{v}\|/\mathrm{d}t$, the derivative of speed with respect to time, which is the same as the scalar component of the acceleration in the direction of the velocity. This is positive if the object is speeding up and negative if the object is slowing down, or decelerating. Section 12.5 of the textbook discusses all of this in detail.)

Reversing this, if you take the indefinite integral of a vector, then the result may be either a point *or* a vector, because differentiating either of these yields a vector. This ambiguity is packaged into the constant of integration. For example, $\int \langle 2t, 3\rangle\,\mathrm{d}t = \langle t^2, 3t\rangle + C$, which is a point if C is a point and a vector if C is a vector. (If C is a vector, then you may want to call it $\mathbf{C}$ instead, but that is just a convention, not a requirement.) The definite integral of a vector, however, is always a vector: fundamentally, you get it by adding up infinitely many infinitesimal vectors (or approximate it by adding up a large number of small vectors), and adding up vectors yields a vector; in practice, you usually calculate it by subtracting indefinite integrals, and regardless of whether you view the indefinite integrals as points or as vectors, subtracting them yields a vector. For example, both $\int_{t=0}^{1} \langle 2t, 3 \rangle \, \mathrm{d}t = \langle t^2, 3t \rangle |_{t=0}^{1} = \langle 1, 3 \rangle - \langle 0, 0 \rangle = \langle 1, 3 \rangle$, and $\int_{t=0}^{1} \langle 2t, 3 \rangle \, \mathrm{d}t = (t^2, 3t)|_{t=0}^{1} = (1, 3) - (0, 0) = \langle 1, 3 \rangle$ give the same result. In fact, either of them could be packaged up as

$$\int_{t=0}^{1} \langle 2t, 3 \rangle \, \mathrm{d}t = \left\langle \int_{t=0}^{1} 2t \, \mathrm{d}t, \int_{t=0}^{1} 3 \, \mathrm{d}t \right\rangle = \left\langle t^2|_{t=0}^{1}, 3t|_{t=0}^{1} \right\rangle = \langle 1 - 0, 3 - 0 \rangle = \langle 1, 3 \rangle.$$

Putting this all together, consider the initial-value problem in which the acceleration of an object is $-32\mathbf{k} = \langle 0, 0, -32 \rangle$ (which is the acceleration of a freely falling object near Earth's surface, if we use units of feet and seconds), the object's initial velocity is $\langle 3, 0, 4 \rangle$ (so a speed of $5\,\mathrm{ft/s}$ eastward and upward with a slope of $4/3$), and the object's initial position is $(0, 0, 100)$ (so 100 feet above the origin on the ground). Then you can calculate a general formula for the object's position P as a function of the elapsed time t by integrating:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \langle 0, 0, -32 \rangle; \\
\mathrm{d}\mathbf{v} &= \langle 0, 0, -32 \rangle \, \mathrm{d}t; \\
\int_{\mathbf{v}=\langle 3,0,4 \rangle} \mathrm{d}\mathbf{v} &= \int_{t=0} \langle 0, 0, -32 \rangle \, \mathrm{d}t; \\
\mathbf{v} - \langle 3, 0, 4 \rangle &= \langle 0, 0, -32t \rangle - \langle 0, 0, -32(0) \rangle; \\
\mathbf{v} &= \langle 3, 0, 4 \rangle + \langle 0, 0, -32t \rangle; \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \langle 3, 0, 4 - 32t \rangle; \\
\mathrm{d}P &= \langle 3, 0, 4 - 32t \rangle \, \mathrm{d}t; \\
\int_{P=(0,0,100)} \mathrm{d}P &= \int_{t=0} \langle 3, 0, 4 - 32t \rangle \, \mathrm{d}t; \\
P - (0, 0, 100) &= \langle 3t, 0, 4t - 16t^2 \rangle - \left\langle 3(0), 0, 4(0) - 16(0)^2 \right\rangle; \\
P &= (0, 0, 100) + \langle 3t, 0, 4t - 16t^2 \rangle; \\
P &= (3t, 0, 100 + 4t - 16t^2).
\end{aligned}$$

In other words, the position after t seconds is $3t$ feet east of the origin at a height of $100 + 4t - 16t^2$ feet.

(In the course of solving this, I've used the *semidefinite integral*:

$$\int_{t=a} f(t) \, \mathrm{d}t = \int_{\tau=a}^{t} f(\tau) \, \mathrm{d}\tau.$$

The Fundamental Theorem of Calculus allows us to calculate these integrals easily:

$$\int_{t=a} F'(t) \, \mathrm{d}t = F(t) - F(a).$$

This is very handy when solving initial-value problems. Since $\mathbf{v} = \langle 3, 0, 4 \rangle$ when $t = 0$, the first step in which I introduced integrals is doing the same operation to both sides of the equation; similarly, the second introduction of integrals is valid because $P = (0, 0, 100)$ when $t = 0$. To solve this problem using indefinite integrals instead requires two extra steps —one for each integration— to find the constants associated with the indefinite integrals, but using semidefinite integrals avoids that.)

Arclength

When finding the length of a curve by integration, you are ultimately integrating an expression such as $\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$. This particular expression applies in 2 dimensions; in words, it is the principal square root of the sum of the square of the differential of x and the square of the differential of y. An expression like this, containing differentials, is called a *differential form*; the textbook mentions differential forms briefly in Section 15.3, but they are really all over the place in this multivariable Calculus, sometimes hidden just under the surface, sometimes out in the open without being acknowledged. I'll be pointing them out whenever they appear.

This particular differential form is called the **arclength element** and is traditionally written $\mathrm{d}s$ (although that notation is misleading for reasons that I will return to later). A simpler way to think of $\mathrm{d}s$, which works in *any* number of dimensions, is as $\|\mathrm{d}P\|$, the magnitude of the differential of the position P. Remember that $\mathrm{d}P$ is a vector when P is a point, so it has a magnitude; in fact, $\mathrm{d}P$ is the same as $\mathrm{d}\mathbf{r}$, so you can also think of $\mathrm{d}s$ as $\|\mathrm{d}\mathbf{r}\|$, the magnitude of the differential of the position vector $\mathbf{r}$. In 2 dimensions, where $P = (x, y)$ and $\mathbf{r} = \langle x, y \rangle$, $\mathrm{d}\mathbf{r} = \mathrm{d}P = \langle \mathrm{d}x, \mathrm{d}y \rangle$, whose magnitude is the arclength element that I talked about above. In 3 dimensions, $DP = \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z \rangle$, whose magnitude is $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$.

When working with a parametrized curve, every variable (x and y, and z if it exists, whether individually or combined into P or $\mathbf{r}$) is given as a function of some parameter t. By differentiating these, their differentials come to be expressed using t and $\mathrm{d}t$. The absolute value $|\mathrm{d}t|$ will naturally appear in the integrand; but if you set up the integral so that t is increasing, then $\mathrm{d}t$ is positive, so $|\mathrm{d}t| = \mathrm{d}t$. Then you can write $\|\mathrm{d}P\|$ as $\|\mathbf{v}\|\,|\mathrm{d}t| = \|\mathbf{v}\|\,\mathrm{d}t$, where $\mathbf{v} = \mathrm{d}P/\mathrm{d}t = \mathrm{d}\mathbf{r}/\mathrm{d}t$ is the velocity, as given in the textbook. More explicitly, this is

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

(in 2 dimensions), which is also given in the textbook. But while you might integrate this in practice to perform a specific calculation, you are most fundamentally integrating a differential form in which t does not appear. This is why the result ultimately does not depend on how you parametrize the curve. (Later on, I'll discuss what it means, in general, to integrate a differential form along a curve, including why and to what extent this is independent of the parametrization.)