Vectors

In this class, we look at spaces with up to 3 dimensions, but most of the ideas in this course (and the next) continue to make sense in spaces with any whole number of dimensions. Although spaces with more than 3 dimensions are difficult to visualize, since we're used to living in a 3-dimensional space, they make perfect sense mathematically. Furthermore, whenever you're trying to keep track of 4 or more independent quantities at once, then you need the mathematics of a space with 4 or more dimensions, whether or not you choose to visualize that space geometrically.

If we assign rectangular coordinates to a space of n dimensions, then the result is called $\mathbf{R}^n$ (or $\mathbb{R}^n$); in particular, a coordinate space of 1 dimension is $\mathbf{R}^1$ or simply $\mathbf{R}$, which is the set of real numbers, or (thinking geometrically) the real number line. You can call the coordinates whatever you like, but it's most common to use x (or sometimes t) as the coordinate in $\mathbf{R}$; then to use x and y as the coordinates in $\mathbf{R}^2$; then x, y, and z in $\mathbf{R}^3$; and finally x_1, x_2, $\ldots$, and x_n in $\mathbf{R}^n$ generally. But there are other systems; as long as you list n independent variables in a row, then you have a valid list of coordinates for $\mathbf{R}^n$.

A **point** in $\mathbf{R}^n$ may be denoted by listing the values of its coordinates in order, separated by commas and optionally surrounded by grouping parentheses. Thus, (x) or (more commonly) x gives a point in the real line $\mathbf{R}$, while (x, y) gives a point in the coordinate plane $\mathbf{R}^2$, (x, y, z) gives a point in the coordinate space $\mathbf{R}^3$, and $(x_1, x_2, \ldots, x_n)$ gives a point in $\mathbf{R}^n$ (which is the most general case).

Sometimes it's nice to have a way to refer to a point in any number of dimensions without having to write a long list with dots in it; then I usually write P for the point. Thus, in 1 dimension, $P = x$; in 2 dimensions, $P = (x, y)$; in 3 dimensions, $P = (x, y, z)$; and in n dimensions, $P = (x_1, x_2, \ldots, x_n)$. So for example, if I say that $P = (2, 3, 5)$, then this is the same as saying that $x = 2$, $y = 3$, and $z = 5$.

It's traditional to use uppercase letters to name points, as I just did. Another tradition is to leave out the equality sign when naming points; so instead of writing $P = (2, 3, 5)$ as I did above, people often just write $P(2, 3, 5)$. I think that this is a terrible convention, so I won't follow it, but you will see it sometimes, even in the textbook.

Vectors

A **vector** is a movement between points. For example, to move in the plane from the point $(2, 3)$ to the point $(3, 1)$, you move 1 unit to the right (in the positive x direction) and 2 units downwards (in the negative y direction). This movement —1 unit to the right and 2 units downwards— is a vector.

A vector in $\mathbf{R}^n$ has the same amount of information as a point there: n real numbers. For this reason, people sometimes write a vector using the same notation as they use to write a point. For example, the vector from the previous paragraph could be written as $(1, -2)$, the same notation as used for the point $(1, -2)$. When referring to a vector, $(1, -2)$ means a movement 1 unit to the right and 2 units downwards; when referring to a point, $(1, -2)$ means the point that lies 1 unit to the right and 2 units downwards from the origin.

However, a vector is not the same thing as a point, and so people often use different notation instead. Common notations for the vector that I've been talking about include $[1, -2]$ and $\langle 1, -2 \rangle$. I will use the last of these, since that is used in the textbook. (There is another notation, which the book uses even more often than $\langle 1, -2 \rangle$, and that is $\mathbf{i} - 2\mathbf{j}$. However, I'll save that for later.) The terminology for these numbers is also different; while 1 and -2 are the *coordinates* of the point $(1, -2)$, we say that 1 and -2 are the **components** of the vector $\langle 1, -2 \rangle$.

Whereas a point tells you a location, a vector tells you only about the motion and nothing about the location. So the vector from $(2, 3)$ to $(3, 1)$ is the same vector as, say, the vector from $(-2, 7)$ to $(-1, 5)$. In both cases, the motion is 1 unit to the right and 2 units downwards, so the vector is $\langle 1, -2 \rangle$.

Motion on a number line corresponds arithmetically to addition. For example, if you start at the number 2 on a number line and move 4 units to the right, then you end up at the number 6, and we represent this fact in arithmetic as $2 + 4 = 6$. Similarly, if you start at $(2, 3)$ and move according to the vector $\langle 1, -2 \rangle$, then you end up at $(3, 1)$, and we represent this fact in arithmetic as $(2, 3) + \langle 1, -2 \rangle = (3, 1)$. So you can add a point and a vector to get another point. Or from another perspective, we could write $6 - 2 = 4$, and similarly $(3, 1) - (2, 3) = \langle 1, -2 \rangle$. So one way to describe a vector is to say that it's what you get when you subtract two points. The textbook doesn't talk about arithmetic with points and vectors like this; it does talk about calculating the vector from one point to another or calculating the point reached from another point by following a given vector, but it doesn't refer to these operations as subtraction and addition. Nonetheless, that's exactly what they are.

The rules for these calculations are very straightforward: you add or subtract corresponding coordinates and components. That is, to get the first coordinate of the sum, you add the first coordinate of the original point and the first component of the vector, and similarly for the second coordinate; or when you subtract two vectors, you subtract the first coordinates of the two points to get the first component of the difference, and similarly for the second component. So you can write out the calculations in full thus:

$$(2,3) + \langle 1,-2\rangle = (2+1, 3-2) = (3,1);$$
$$(3,1) - (2,3) = \langle 3-2, 1-3\rangle = \langle 1,-2\rangle.$$

Here are general formulas for this rule in any number of dimensions:

$$(a_1, a_2, \ldots, a_n) + \langle v_1, v_2, \ldots, v_n\rangle = (a_1 + v_1, a_2 + v_2, \ldots, a_n + v_n);$$
$$(b_1, b_2, \ldots, b_n) - (a_1, a_2, \ldots, a_n) = \langle b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n\rangle.$$

When I use P to denote a generic point, I'll use ΔP to denote a generic vector. Here, the uppercase Greek letter Delta, 'Δ', which stands for 'difference', is commonly used to indicate the amount by which the value of some quantity changes. (Think of $\Delta y/\Delta x$ for the slope of a line.) That is,

$$\Delta P = P_1 - P_0,$$

or

$$\Delta P = \langle \Delta x_1, \Delta x_2, \ldots, \Delta x_n\rangle.$$

When you give a vector a name of its own, however, it's common to use a boldface lowercase letter, such as $\mathbf{u}$ or $\mathbf{v}$. Thus, if I use $\mathbf{v}$ to refer to the vector that I've been using as an example throughout this section, then I would write $\mathbf{v} = \langle 1,-2\rangle$. In handwriting, you can write a little arrow over the letter instead, to produce something like $\vec{v}$; other common conventions are to underline or overline vectors, producing symbols such as $\underline{v}$ or $\overline{v}$. On the other hand, it's OK to just write v if you want. The meaning of any symbol that you use should be clear from the context that you provide; in particular, the context should make clear whether a symbol refers to a number, function, point, vector, or whatever, regardless of whatever fancy fonts or decorations you may or may not use.

Arithmetic with vectors

Besides adding vectors to points and subtracting points to get a vector, you can also do arithmetic within the world of vectors itself. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in n dimensions, both representing some motion within $\mathbf{R}^n$, then $\mathbf{u} + \mathbf{v}$ represents the motion of $\mathbf{u}$ followed by the motion of $\mathbf{v}$. This is consistent with how addition of motions works on a number line; for example, if you move 4 units to the right and then move 3 units to the right, then overall you're moving $4 + 3 = 7$ units to the right.

If $\mathbf{v}$ is a vector, then $-\mathbf{v}$ is the vector representing the opposite motion. Again, this matches arithmetic on a number line; the opposite of moving 4 units to the right is moving 4 units to the left, which is represented by the number -4. Then $\mathbf{u} - \mathbf{v}$ just means $\mathbf{u} + (-\mathbf{v})$.

You calculate these by the same principles as arithmetic between points and vectors. For example, to add $\langle 1,-2\rangle$ and $\langle 3,5\rangle$, you simply add the corresponding components:

$$\langle 1,-2\rangle + \langle 3,5\rangle = \langle 1+3, -2+5\rangle = \langle 4,3\rangle.$$

And this should make sense; if you move 1 unit to the right and 2 units downwards, then move 3 units to the right and 5 units upwards, then overall you're moving 4 units to the right and 3 units upwards. Similarly,

$$\langle 1,-2\rangle - \langle 3,5\rangle = \langle 1-3, -2-5\rangle = \langle -2,-7\rangle.$$

That is, if you move 1 unit to the right and 2 units downards and then move the opposite of 3 units to the right and 5 units upwards (which is 3 units to the left and 5 units downwards), then overall you're moving 2 units to the left and 7 units downwards. Here are the general formulas in $\mathbf{R}^n$:

$$\begin{aligned}\langle u_1, u_2, \ldots, u_n\rangle + \langle v_1, v_2, \ldots, v_n\rangle &= \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n\rangle;\\ \langle u_1, u_2, \ldots, u_n\rangle - \langle v_1, v_2, \ldots, v_n\rangle &= \langle u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n\rangle.\end{aligned}$$

Besides adding and subtracting vectors, you can multiply or divide them by real numbers. For example, if $\mathbf{v}$ is a vector representing some motion, then $2\mathbf{v}$ represents doing that motion twice, $1/2\,\mathbf{v}$ or $\mathbf{v}/2$ represents performing half of that motion, $-2\mathbf{v}$ represents making the opposite motion twice, and so on. You calculate these by multiplying each component by that same real number; for example,

$$\begin{aligned}2\langle 1, -2\rangle &= \langle 2(1), 2(-2)\rangle = \langle 2, -4\rangle,\\ \frac{1}{2}\langle 1, -2\rangle &= \left\langle \frac{1}{2}(1), \frac{1}{2}(-2)\right\rangle = \left\langle \frac{1}{2}, -1\right\rangle \text{ or}\\ \frac{\langle 1, -2\rangle}{2} &= \left\langle \frac{1}{2}, \frac{-2}{2}\right\rangle = \left\langle \frac{1}{2}, -1\right\rangle, \text{ and}\\ -2\langle 1, -2\rangle &= \langle -2(1), -2(-2)\rangle = \langle -2, 4\rangle.\end{aligned}$$

Here are the general formulas in $\mathbf{R}^n$:

$$\begin{aligned}a\langle v_1, v_2, \ldots, v_n\rangle &= \langle av_1, av_2, \ldots, av_n\rangle;\\ \frac{\langle v_1, v_2, \ldots, v_n\rangle}{a} &= \left\langle \frac{v_1}{a}, \frac{v_2}{a}, \ldots, \frac{v_n}{a}\right\rangle \text{ for } a \neq 0.\end{aligned}$$

This operation is called **scalar multiplication** (or *scalar division*), because geometrically it amounts to changing the scale used to measure the vector (at least when the real number in question is positive). As a result of this, numbers are often called **scalars** when working with vectors, even though the word 'number' would work perfectly well.

More generally, you can take any homogeneous linear expression (that is a linear expression without a constant term) in any number of variables, replace the variables with vectors, and get a legitimate operation on vectors. Such an operation is called, in general, a **linear combination**. For example, $2\mathbf{u} + 3\mathbf{v} - 5\mathbf{w}$ is a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Geometrically, this represents moving twice according to $\mathbf{u}$, then moving 3 times according to $\mathbf{v}$, and moving 5 times the reverse of the motion given by $\mathbf{w}$.

Still more generally, you can replace the variables with points or vectors; if the sum of the coefficients on the points is 0, then the result is a vector, and if the sum of the coefficients on the points is 1, then the result is a point. For example, if A, B, and C are points, while $\mathbf{u}$ and $\mathbf{v}$ are vectors, then $2A - 3B + 2C + 4\mathbf{u} - 5\mathbf{v}$ is a point (because $2 - 3 + 2 = 1$), while $2A - 3B + C + 4\mathbf{u} - 5\mathbf{v}$ is a vector (because $2 - 3 + 1 = 0$). Geometrically, $2A - 3B + 2C + 4\mathbf{u} - 5\mathbf{v}$ means the point that you reach by starting at A, moving as you would move to get to A from B, then moving twice as you would move to get to C from B, then moving 4 times according to $\mathbf{u}$, and moving 5 times the reverse of the motion given by $\mathbf{v}$. (That is, think of it as $A + (A - B) + 2(C - B) + 4\mathbf{u} - 5\mathbf{v}$.) Similarly, $2A - 3B + C + 4\mathbf{u} - 5\mathbf{v}$ is the motion consisting of moving twice as you would move to get to A from B, then moving as you would move to get to C from B, then moving 4 times according to $\mathbf{u}$, and moving 5 times the reverse of the motion given by $\mathbf{v}$. (That is, think of it as $2(A - B) + (C - B) + 4\mathbf{u} - 5\mathbf{v}$.)

Another example of a point is $1/3\,A + 1/3\,B + 1/3\,C$, which is the average of the 3 points. If you think of this as $A + 2/3\,(B - A) + 1/3\,(C - B)$, then you can describe this in terms similar to those of the previous examples, but in this case it's probably better to think of it directly as an average.

If the sum of the coefficients on the points is neither 1 nor 0, then there is no direct geometric interpretation of the linear combination, but you can still perform calculations with such things; they basically represent internal parts of a larger calculation, such as the $2A - 3B$ that begins some of the examples above.

All of the usual algebraic identities apply to linear combinations of points and vectors. For example, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$, $(A+\mathbf{u})+\mathbf{v}=A+(\mathbf{u}+\mathbf{v})$, $2(\mathbf{u}+\mathbf{v})=2\mathbf{u}+2\mathbf{v}$, and so on. Although you can prove these geometrically, the simplest way to verify them is to do so component by component; then they reduce to identities about real numbers.

You could try multiplying and dividing vectors by each other using the same method of calculation as you use for adding and subtracting them, component by component. People do this sometimes, but there's no geometric interpretation of these operations, neither directly nor as part of a larger calculation with a geometric interpretation. So we won't be doing that. Instead, we'll see some other methods of multiplying vectors later on.

The **zero vector**, denoted $\mathbf{0}$, represents no motion at all. It's general formula in $\mathbf{R}^n$ is

$$\mathbf{0}=\langle 0,0,\ldots,0\rangle.$$

It obeys algebraic rules analogous to those obeyed by the real number 0, such as $\mathbf{0}+\mathbf{v}=\mathbf{v}$, $\mathbf{v}-\mathbf{v}=\mathbf{0}$, and $A+\mathbf{0}=A$. (The last of these demonstrates what it means to say that $\mathbf{0}$ represents no motion at all; you start at the point A, do nothing, and wind up still at A.)

The standard basis vectors

There are some other special symbols for special vectors, and these lead to another general system of notation for vectors (and points).

In $\mathbf{R}^2$, there are 2 **standard basis vectors**, $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{i}=\langle 1,0\rangle,\ \mathbf{j}=\langle 0,1\rangle.$$

In $\mathbf{R}^3$, there are 3 of them:

$$\mathbf{i}=\langle 1,0,0\rangle,\ \mathbf{j}=\langle 0,1,0\rangle,\ \mathbf{k}=\langle 0,0,1\rangle.$$

In $\mathbf{R}^n$, there is a shift in the usual notation:

$$\mathbf{e}_1=\langle 1,0,0,\ldots,0\rangle,\ \mathbf{e}_2=\langle 0,1,0,0,\ldots,0\rangle,\ \ldots,\ \mathbf{e}_n=\langle 0,0,\ldots,0,0,1\rangle.$$

The value of this is that any vector can be written as a unique linear combination of the standard basis vectors:

$$\begin{aligned}\langle a,b\rangle&=a\mathbf{i}+b\mathbf{j};\\ \langle a,b,c\rangle&=a\mathbf{i}+b\mathbf{j}+c\mathbf{k};\\ \langle a_1,a_2,\ldots,a_n\rangle&=a_1\mathbf{e}_1+a_2\mathbf{e}_2+\cdots+a_n\mathbf{e}_n.\end{aligned}$$

Work out the right-hand sides of these and see for yourself that you get the left-hand side. (It's a little annoying that $\mathbf{i}$ and $\mathbf{j}$ are ambiguous, but as long as you know whether they're supposed to be in $\mathbf{R}^2$ or in $\mathbf{R}^3$, then you know what they mean.)

If a component of a vector happens to be 1, then you can leave it out of the expression in the standard basis vectors; if the component is negative, then you use subtraction instead of addition; if the component is 0, then you leave that term out entirely. For example, $\langle 1,-2\rangle=1\mathbf{i}+(-2)\mathbf{j}=\mathbf{i}-2\mathbf{j}$. In $\mathbf{R}^3$, $\langle 1,-2,0\rangle$ is also written $\mathbf{i}-2\mathbf{j}$, because the component on $\mathbf{k}$ is 0.

You can now do arithmetic with vectors by following the ordinary rules of algebra and leaving the symbols for the standard basis vectors alone. For example, instead of $\langle 1,-2\rangle+\langle 3,5\rangle=\langle 4,3\rangle$, you calculate

$$(\mathbf{i}-2\mathbf{j})+(3\mathbf{i}+5\mathbf{j})=(1+3)\mathbf{i}+(-2+5)\mathbf{j}=4\mathbf{i}+3\mathbf{j}.$$

Similarly, instead of $2\langle 1,-2\rangle=\langle 2,-4\rangle$, you calculate

$$2(\mathbf{i}-2\mathbf{j})=2\mathbf{i}-2(2\mathbf{j})=2\mathbf{i}-4\mathbf{j}.$$

You can even extend this notation to points by introducing O for the origin of the coordinate system. That is,

$$O=(0,0,\ldots,0)$$

in $\mathbf{R}^n$. Then any point can be described by starting at the origin and moving along a vector whose components are the coordinates of that point; for example, $(2,3) = O + \langle 2,3 \rangle = O + 2\mathbf{i} + 3\mathbf{j}$. Then you can again do calculations using the rules of algebra; for example, instead of $(2,3) + \langle 1,-2 \rangle = (3,1)$, you calculate

$$(O + 2\mathbf{i} + 3\mathbf{j}) + (\mathbf{i} - 2\mathbf{j}) = O + (2+1)\mathbf{i} + (3-2)\mathbf{j} = O + 3\mathbf{i} + \mathbf{j}.$$

The textbook uses this notation for vectors most of the time, although it continues to use a list of coordinates with commas for points, which it has to do since it never refers directly to addition of points and vectors.

Lengths and angles

In many situations, we want to refer to the distance between two points, or equivalently to the length of a vector. This goes by several names; in general, the **length**, **magnitude**, or **norm** of a vector in $\mathbf{R}^n$ is

$$\|\langle v_1, v_2, \ldots, v_n \rangle\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}.$$

Here I've denoted the length of a vector $\mathbf{v}$ as $\|\mathbf{v}\|$, although the textbook writes this as simply $|\mathbf{v}|$ instead. As a statement about distances, this is the n-dimensional generalization of the Pythagorean Theorem.

One basic algebraic property of lengths is

$$\|a\mathbf{v}\| = |a| \, \|\mathbf{v}\|.$$

(Note that you must write $|a|$ when a is a scalar, even if you choose to use the notation $\|\mathbf{v}\|$ when $\mathbf{v}$ is a vector.) You can check this from the general formula by factoring inside the square root; remember the identity $\sqrt{a^2} = |a|$ for arbitrary real numbers. (It's a common algebra mistake to think that $\sqrt{a^2} = a$; this is correct when $a \geq 0$ but not otherwise.) In particular,

$$\|-\mathbf{v}\| = \|\mathbf{v}\|.$$

Also,

$$\|\mathbf{0}\| = 0;$$

conversely, if $\|\mathbf{v}\| = 0$, then it must be that $\mathbf{v} = \mathbf{0}$. (Ultimately this is because a sum of squares of real numbers can only be zero if all of the original numbers are zero.)

There is no general formula for $\|\mathbf{u} + \mathbf{v}\|$; however, we can say

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

This is called the **triangle inequality**, since if you draw a triangle whose sides are $\mathbf{u}$, $\mathbf{v}$, and their sum $\mathbf{u} + \mathbf{v}$, then this expresses the fact that the length of the last side is the shortest distance between its two endpoints. You can check this from the formula by squaring both sides, cancelling some common terms, squaring again, subtracting the two sides, and factoring the result as a perfect square. You can then argue that this perfect square is greater than or equal to zero, so the right-hand side just before the subtraction is greater than or equal to the left-hand side at that stage, and this remains so upon taking principal square roots, adding some common terms, and taking principal square roots again. I'll skip the details.

If $\mathbf{v} \neq \mathbf{0}$ (so that you can divide by $\|\mathbf{v}\|$), then $\mathbf{v}/\|\mathbf{v}\|$ is a vector whose own magnitude is 1. (This is because

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \frac{\|\mathbf{v}\|}{|\|\mathbf{v}\||} = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1,$$

using that $\|\mathbf{v}\| \geq 0$.) This is called the **unit vector** in the direction of $\mathbf{v}$, or simply the **direction** of $\mathbf{v}$. The usual notation for this is $\hat{\mathbf{v}}$:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

For some reason, the book never introduces this notation (or any other notation for this concept), but it refers to the idea itself quite often. Notice that you can write $\mathbf{v} = \|\mathbf{v}\|\hat{\mathbf{v}}$; this expresses the common slogan that a vector has both a length and a direction. (However, the zero vector has only a length, of 0, and no way to pick out any unit vector as its direction.)

If you perform some algebraic tricks with the triangle inequality and assume that neither $\mathbf{u}$ nor $\mathbf{v}$ is the zero vector $\mathbf{0}$ (so that you can divide by their norms), then you can also derive the compound inequality

$$-1 \leq \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{2\,\|\mathbf{u}\|\,\|\mathbf{v}\|} \leq 1.$$

(I'll skip this derivation, but it's based on first replacing $\mathbf{v}$ with $-\mathbf{v}$, squaring both sides, and rearranging terms to derive one half of this result, then going back to the beginning and replacing $\mathbf{u}$ with $\mathbf{u} - \mathbf{v}$, squaring both sides again, and rearranging terms to derive the other half of the result.) If you draw a triangle whose sides are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ (so that $\mathbf{u}$ and $\mathbf{v}$ are both starting from the same point), then the Law of Cosines says that the expression in the middle of the compound inequality above is the cosine of the angle between the sides $\mathbf{u}$ and $\mathbf{v}$, and the inequality verifies that this lies within the possible range of values for a cosine. (If either $\mathbf{u}$ or $\mathbf{v}$ is the zero vector, then you don't really have a triangle, and this angle doesn't make sense.)

If you have two rays emanating from the same point in a multidimensional space, then the only way to describe the angle between them is with an angle between 0 and π (or 180°), which is the range of possible values of an arccosine (or inverse cosine), so taking the arccosine of the expression above gives you this angle:

$$\angle(\mathbf{u}, \mathbf{v}) = \operatorname{acos}\left(\frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{2\,\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

(In $\mathbf{R}^2$, and only in $\mathbf{R}^2$, it's possible to distinguish clockwise and counterclockwise angles, which I'll come back to when I discuss the cross product below.) Thus, it's possible to describe both lengths and angles using vectors, through the concept of the magnitude of a vector. (There's a more efficient way to calculate this cosine, which we'll see later on using the dot product, but it's important that angles can be calculated from lengths alone.)

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **perpendicular** or **orthogonal** if the angle between them is a right angle ($\pi/2$, or 90°), whose cosine is 0; the symbol for this is $\mathbf{u} \perp \mathbf{v}$. Similarly, $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if the angle between them is the zero angle (0, or 0°), whose cosine is 1; the symbol for this is $\mathbf{u} \parallel \mathbf{v}$. However, people sometimes use this symbol (or even the word 'parallel') to include the case where $\mathbf{u}$ and $\mathbf{v}$ are **antiparallel**, meaning that the angle between them is a straight angle (π, or 180°), whose cosine is -1.

However, for many applications of vectors, the concept of length or magnitude really doesn't make sense! This is because vectors describe motion within any space with any coordinates, and those coordinates might refer to incompatible quantities. For example, if x measures time and y measures something that changes with time but is not itself a time (the height of a falling object, the price of a stock, the population of the world, or nearly any other quantity of interest), then it really doesn't make sense to talk about the magnitude

$$\|\Delta P\| = \|\langle \Delta x, \Delta y\rangle\| = \sqrt{\Delta x^2 + \Delta y^2}.$$

You can see this if you imagine what units of measurement you might use for such a magnitude; if x is measured in seconds and y is measured in metres (as one might do when talking about the height of a falling object, for example), then which unit is $\|\Delta P\|$ in? Neither one makes sense, nor does any combination of them.

So while lengths of vectors and angles between them always exist in the realm of mathematical abstraction, they can only be meaningful when all of the coordinates measure the same type of quantity. (Even then, these concepts may or may not really be meaningful, but at least they have a chance.) The exception to this is that we can say whether two nonzero vectors are parallel (or antiparallel) without reference to angles: $\mathbf{u}$ and $\mathbf{v}$ are parallel if there is a scalar $k > 0$ such that $\mathbf{u} = k\mathbf{v}$; they're antiparallel if there is a scalar $k < 0$ such that $\mathbf{u} = k\mathbf{v}$.

Projections

If you have two vectors $\mathbf{u}$ and $\mathbf{v}$, and assuming that neither of them is $\mathbf{0}$, place them so that they both start at the same point A and then draw a line from $A+\mathbf{v}$ to the line through A and $A+\mathbf{u}$ so that these lines intersect at a right angle. Let B be the point where these lines intersect; the vector $B-A$ is the **projection** of $\mathbf{v}$ onto $\mathbf{u}$, denoted $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$. Sometimes people also consider the projection of $\mathbf{v}$ *perpendicular* to $\mathbf{u}$; this is the vector from $A+\mathbf{u}$ to B:

$$\operatorname{proj}_{\mathbf{u}}^{\perp} \mathbf{v} = \mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}.$$

(In general, the symbol '$\perp$' is used when talking about perpendicular things, which the shape of the symbol is supposed to remind you of.)

A related concept is the **component** of $\mathbf{v}$ in the direction of $\mathbf{u}$, denoted $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$; this is a scalar chosen so that

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \operatorname{comp}_{\mathbf{u}} \mathbf{v}\, \hat{\mathbf{u}}.$$

It's a common mistake to think that $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ has the same direction as $\mathbf{u}$, so that consequently $\operatorname{comp}_{\mathbf{u}} \mathbf{v} = \|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|$. But in fact, $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ can just as easily have the opposite direction, so the general rule is

$$|\operatorname{comp}_{\mathbf{u}} \mathbf{v}| = \|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|.$$

The component of $\mathbf{v}$ in the direction of $\mathbf{u}$ is positive if $\mathbf{u}$ and $\mathbf{v}$ have roughly the same direction but negative if they have roughly opposite directions. (It's also possible that this component is zero, when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.)

I have not allowed $\mathbf{v}$ to be the zero vector, because then $A+\mathbf{v}$ is simply A, right on the line through A and $A+\mathbf{u}$, so it makes no sense to draw anything from that point perpendicular to that line. However, since we're already on the line, we can simply take B to be A as well, so that $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$, which is $B-A$, is also $\mathbf{0}$. Thus, we have these results:

$$\operatorname{proj}_{\mathbf{u}} \mathbf{0} = \mathbf{0},\ \operatorname{comp}_{\mathbf{u}} \mathbf{0} = 0.$$

Now $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ and $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$ exist no matter what $\mathbf{v}$ is (although it's still necessary that $\mathbf{u} \neq \mathbf{0}$). Once we have that, you can verify these facts by drawing the relevant pictures:

$$\operatorname{proj}_{\mathbf{u}} (\mathbf{v}+\mathbf{w}) = \operatorname{proj}_{\mathbf{u}} \mathbf{v} + \operatorname{proj}_{\mathbf{u}} \mathbf{w}, \text{ so } \operatorname{comp}_{\mathbf{u}} (\mathbf{v}+\mathbf{w}) = \operatorname{comp}_{\mathbf{u}} \mathbf{v} + \operatorname{comp}_{\mathbf{u}} \mathbf{w};$$
$$\operatorname{proj}_{\mathbf{u}} (a\mathbf{v}) = a \operatorname{proj}_{\mathbf{u}} \mathbf{v}, \text{ so } \operatorname{comp}_{\mathbf{u}} (a\mathbf{v}) = a \operatorname{comp}_{\mathbf{u}} \mathbf{v}.$$

This is all well and good, but if you know a little trigonometry, then you can get a nice formula for this component. This is because $\mathbf{v}$ forms the hypotenuse of a right triangle, one of whose legs is $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$, and whose angle next to that leg is $\angle(\mathbf{u},\mathbf{v})$ if $\mathbf{u}$ and $\mathbf{v}$ have roughly the same direction or $\pi - \angle(\mathbf{u},\mathbf{v})$ if they have roughly opposite directions. In the first case,

$$\cos\angle(\mathbf{u},\mathbf{v}) = \frac{\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|};$$

in the other case,

$$\cos\angle(\mathbf{u},\mathbf{v}) = -\cos\Big(\pi - \angle(\mathbf{u},\mathbf{v})\Big) = -\frac{\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|}{\|\mathbf{v}\|} = -\frac{-\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|} = \frac{\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|}.$$

In the middle, when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $\cos\angle(\mathbf{u},\mathbf{v})$ and $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$ are both 0. So in any case,

$$\operatorname{comp}_{\mathbf{u}} \mathbf{v} = \|\mathbf{v}\| \cos\angle(\mathbf{u},\mathbf{v})$$

as long as $\mathbf{v} \neq \mathbf{0}$. (If $\mathbf{v} = \mathbf{0}$, then the angle $\angle(\mathbf{u},\mathbf{v})$ doesn't make sense, but the equation is still true in a way, since it becomes the true statement $0 = 0$ no matter what value you use for the angle.) We saw earlier how to express this cosine using only $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u}-\mathbf{v}\|$, but for now, let's just leave it as $\cos\angle(\mathbf{u},\mathbf{v})$.

The dot product

This now suggests that we'll get a very nice operation if we define

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \operatorname{comp}_{\mathbf{u}} \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \angle(\mathbf{u}, \mathbf{v}).$$

This has many nice properties; for example, these follow from the corresponding properties for components:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w}),$$
$$\mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v}).$$

However, since $\mathbf{u}$ and $\mathbf{v}$ appear symmetrically in the formula with the cosine, we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u},$$

and then these properties also follow:

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w}),$$
$$(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}).$$

The definition $\|\mathbf{u}\| \operatorname{comp}_{\mathbf{u}} \mathbf{v}$ allows $\mathbf{v}$ to be $\mathbf{0}$, but not $\mathbf{u}$. However, since the operation is symmetric when the vectors are nonzero, we can define it so that it continues to be symmetric, so that $\mathbf{0} \cdot \mathbf{v} = 0$ as well as $\mathbf{v} \cdot \mathbf{0} = 0$. In particular, we define $\mathbf{0} \cdot \mathbf{0}$ to be 0. (Thus, it remains true in a way that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \angle(\mathbf{u}, \mathbf{v})$, even when $\angle(\mathbf{u}, \mathbf{v})$ doesn't make sense, because in that case the equation becomes $0 = 0$ no matter what value you use for the angle.) Then the properties listed above continue to be true.

By this point, you should see where the notation comes from; this operation has a lot of the same properties as multiplication. It's variously called **inner multiplication** (for the operation) or the **inner product** (for the result of the operation), the **scalar product** (because the result is a scalar), or (naming it after its notation) the **dot product**. (Don't confuse *scalar multiplication*, describing the operation for $a\mathbf{v}$, with the *scalar product*, describing the result of the operation $\mathbf{u} \cdot \mathbf{v}$.) The properties above state that the dot product distrubutes over addition, that it's commutative, associative with scalar multiplication, etc.

Since angles can be expressed in terms of lengths, so can the dot product; you get

$$\mathbf{u} \cdot \mathbf{v} = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{2},$$

an expression that works regardless of whether $\mathbf{u}$ and $\mathbf{v}$ are nonzero. An important special case is when $\mathbf{u}$ and $\mathbf{v}$ are the same vector; then this simplifies to

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2.$$

(Another way to see this is that the angle between a vector and itself is 0, the cosine of which is 1, so $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\| \, \|\mathbf{v}\| \cos 0 = \|\mathbf{v}\|^2$.)

However, as a practical matter, there is a better way to calculate this. Because the dot product distributes over addition and associates with scalar multiplication, we only need to know $\mathbf{i} \cdot \mathbf{i}$, $\mathbf{i} \cdot \mathbf{j}$, and so on; that is, we only need to know what it does to the standard basis vectors. Since these vectors are all perpendicular to one another, so the cosine between any two different ones is 0, these dot products are almost all 0. The exception is the dot product of one of these with itself; since these vectors all have a magnitude of 1, the dot product of any one with itself is $1^2 = 1$. So in 2 dimensions,

$$\langle a, b \rangle \cdot \langle c, d \rangle = (a\mathbf{i} + b\mathbf{j}) \cdot (c\mathbf{i} + d\mathbf{j}) = ac\,\mathbf{i} \cdot \mathbf{i} + ad\,\mathbf{i} \cdot \mathbf{j} + bc\,\mathbf{j} \cdot \mathbf{i} + bd\,\mathbf{j} \cdot \mathbf{j} = ac\,1 + ad\,0 + bc\,0 + bd\,1 = ac + bd;$$

in 3 dimensions,

$$\langle a, b, c\rangle \cdot \langle d, e, f\rangle = ad + be + cf$$

by a similar calculation, and most generally in n dimensions,

$$\langle a_1, a_2, \ldots, a_n\rangle \cdot \langle b_1, b_2, \ldots, b_n\rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

That is, you multiply corresponding components of the vectors and add these all up. For example,

$$\langle 1, -2\rangle \cdot \langle 3, 5\rangle = (1)(3) + (-2)(5) = 3 - 10 = -7.$$

Now its best to give formulas for angles, projections, and components in terms of the dot product, rather than the other way around. So:

$$\begin{aligned}
\operatorname{comp}_{\mathbf{u}} \mathbf{v} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}; \\
\operatorname{proj}_{\mathbf{u}} \mathbf{v} &= \operatorname{comp}_{\mathbf{u}} \mathbf{v}\, \hat{\mathbf{u}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}; \\
\angle(\mathbf{u}, \mathbf{v}) &= \operatorname{acos} \frac{\operatorname{comp}_{\mathbf{u}} \mathbf{v}}{\|\mathbf{v}\|} = \operatorname{acos} \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\, \|\mathbf{v}\|}.
\end{aligned}$$

Even lengths can be expressed using the dot product:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

Row vectors

I developed the dot product geometrically, and we've seen that it's closely related to lengths and angles. I remarked before that lengths and angles don't always make sense, and the same goes for the the dot product (as well as projections and components onto a given vector). For example, if x is measured in seconds (s) and y is measured in metres (m), then $\langle 1\,\mathrm{s}, -2\,\mathrm{m}\rangle \cdot \langle 3\,\mathrm{s}, 5\,\mathrm{m}\rangle = 3\,\mathrm{s}^2 - 10\,\mathrm{m}^2$ doesn't really make sense.

On the other hand, sometimes dot products can make sense in a context like this. For example, suppose that x represents the time at which something occurs and y represents its location, so that the vector $\Delta P = \langle \Delta x, \Delta y\rangle$ represents a passage of time together with a change in location, like the vectors above might do; then if the object in question is a missile that's going to explode at some unkown time and distance and you think that it's going to move slowly while I think that it's going to move quickly, then we might make a bet where I pay you \$1 for every second that it lasts until it explodes but you pay me \$2 for every metre that it travels. If it travels 5 metres in 3 seconds before exploding, then you'll get $(1)(3) - (2)(5) = -7$ dollars, or put another way, you'll owe me \$7. This can be represented as the dot product

$$\langle \$1/\mathrm{s}, -\$2/\mathrm{m}\rangle \cdot \langle 3\,\mathrm{s}, 5\,\mathrm{m}\rangle = (\$1/\mathrm{s})(3\,\mathrm{s}) + (-\$2/\mathrm{m})(5\,\mathrm{m}) = \$3 - \$10 = -\$7,$$

where the first vector is determined by the nature of our bet (you get \$1 per second and pay \$2 per metre), while the second vector is determined by the behaviour of the missile (it lasts 3 seconds and travels 5 metres).

Now, while the vector $\langle 3\,\mathrm{s}, 5\,\mathrm{m}\rangle$ really does describe a change in x and a change in y, where x and y represent time and position as I stated above, the vector $\langle \$1/\mathrm{s}, -\$2/\mathrm{m}\rangle$ does not. In the context of measuring time and position, this vector is a different kind of vector, one for which a dot product with an ordinary vector makes sense, even though lengths and angles don't make sense for any of these vectors. A vector like this is variously called a **dual vector**, a **covector**, or a **row vector**; in the last case, an ordinary vector may be called a **column vector**. I'll use the terminology of row and column vectors, which ultimately comes from matrix theory.

Row vectors obey the same rules of arithmetic as column vectors; here is a list of operations with these that make sense:

- Addition: adding a column vector to a point to get another point, adding two column vectors together to get another column vector, adding two row vectors together to get another row vector;
- Subtraction: subtracting a column vector from a point to get another point, subtracting one column vector from another to get another column vector, subtracting one row vector from another to get another row vector;
- Multiplication: multiplying a column vector by a scalar to get another column vector, multiplying a row vector by a scalar to get another row vector, multiplying a row vector and a column vector to get a scalar.

In particular, there is (in general) no notion of 'row point' that can interact with row vectors in the way that points interact with column vectors.

Area

Now let's go back to a geometric conception of vectors. If you take two vectors $\mathbf{u}$ and $\mathbf{v}$ and place them to start at a point A, then you can connect their endpoints to make a triangle and then ask what the area of that triangle is. It's actually a bit nicer to think of that triangle as half of a parallelogram: two opposite sides of the parallelogram are $\mathbf{u}$, one running from A to $A+\mathbf{u}$, the other running from $A+\mathbf{v}$ to $A+\mathbf{v}+\mathbf{u}$; the other two opposite sides are $\mathbf{v}$, one running from A to $A+\mathbf{v}$, the other running from $A+\mathbf{u}$ to $A+\mathbf{u}+\mathbf{v}$ (which of course is the same as $A+\mathbf{v}+\mathbf{u}$).

This question can be asked in any number of dimensions, and the answer may be written $\|\mathbf{u}\times\mathbf{v}\|$. This notation suggests that this area will be the magnitude of something more fundamental, which is $\mathbf{u}\times\mathbf{v}$ itself, and this is true to an extent, but exactly how that works depends on how many dimensions we're in. So for now, I'm just going to stick with $\|\mathbf{u}\times\mathbf{v}\|$. However, I can give you the terminology: whatever $\mathbf{u}\times\mathbf{v}$ is, the operation may be called **outer multiplication**, and the result may be called the **outer product** or the **cross product**, and in 3 dimensions (where it is best known), it's also called the **vector product**.

With the help of trigonometry,

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\angle(\mathbf{u},\mathbf{v}).$$

Notice that this sine is always positive, since the angle lies between 0 and π. For such an angle θ, $\sin\theta = \sqrt{1-\cos^2\theta}$; with the help of the dot product, this means that

$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2}.$$

(This formula makes sense even if $\mathbf{u}$ or $\mathbf{v}$ is the zero vector, in which case the result is zero.) If you write out $\mathbf{u}\cdot\mathbf{v}$ in this expression in terms of the lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u}-\mathbf{v}\|$, then the formula factors as

$$\|\mathbf{u}\times\mathbf{v}\| = \frac{\sqrt{-(\|\mathbf{u}\|+\|\mathbf{v}\|+\|\mathbf{u}-\mathbf{v}\|)(\|\mathbf{u}\|+\|\mathbf{v}\|-\|\mathbf{u}-\mathbf{v}\|)(\|\mathbf{u}\|-\|\mathbf{v}\|+\|\mathbf{u}-\mathbf{v}\|)(\|\mathbf{u}\|-\|\mathbf{v}\|-\|\mathbf{u}-\mathbf{v}\|)}}{2}.$$

(Despite the initial minus sign, the expression inside the square root is positive, since the last factor is negative.) This result was known to the ancient Greek–Egyptian mathematician and inventor Hero (or Heron) of Alexandria. (He invented the steam engine, the windmill, and the vending machine, although none of those caught on at the time.)

If $\mathbf{u}$ and $\mathbf{v}$ are parallel (or antiparallel), or if either (or both) of them is the zero vector $\mathbf{0}$, then $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\,\|\mathbf{v}\|$, so $\|\mathbf{u}\times\mathbf{v}\| = 0$. From another perspective, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, then the angle between them is 0, whose sine is 0; if they're antiparallel, then the sine is still $\sin\pi = 0$. In this case, you don't really have a parallelogram, but a simple line segment (or a point if $\mathbf{u}$ and $\mathbf{v}$ are both $\mathbf{0}$), whose area is indeed zero.

Here are some important algebraic properties of $\|\mathbf{u}\times\mathbf{v}\|$:

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{v}\times\mathbf{u}\|;$$
$$\|\mathbf{u}\times a\mathbf{v}\| = |a|\,\|\mathbf{u}\times\mathbf{v}\|;$$
$$\|\mathbf{u}\times\mathbf{v}\| = \left\|\mathbf{u}\times\operatorname{proj}^{\perp}_{\mathbf{u}}\mathbf{v}\right\| = \|\mathbf{u}\|\left\|\operatorname{proj}^{\perp}_{\mathbf{u}}\mathbf{v}\right\|.$$

(The last of these assumes that $\mathbf{u}\neq\mathbf{0}$, so that projection perpendicular to $\mathbf{u}$ makes sense.) These should be obvious geometrically; in particular, the last of these states that the area of a parallelogram is the same as the area of a rectangle with the same base and height.

The cross product in three dimensions

For vectors in $\mathbf{R}^3$, we can interpret $\mathbf{u} \times \mathbf{v}$ as a vector. The magnitude $\|\mathbf{u} \times \mathbf{v}\|$ is the area from the previous section, so we only need to describe the direction of $\mathbf{u} \times \mathbf{v}$: it will be perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Most of the time, there are precisely two directions perpendicular to two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$. To decide which of these is the direction of $\mathbf{u} \times \mathbf{v}$, we use the *right-hand rule*: if you start by pointing the fingers of your right hand in the direction of $\mathbf{u}$, curl them to point in the direction of $\mathbf{v}$, and then stick out your thumb, then your thumb will point roughly in the direction of $\mathbf{u} \times \mathbf{v}$. (This should be used together with a right-handed coordinate system: if you point your fingers along the positive x-axis, curl them to point along the positive y-axis, and then stick out your thumb, then your thumb will point roughly along the positive z-axis.) If $\mathbf{u}$ and $\mathbf{v}$ happen to be parallel (or antiparallel), or if either (or both) of them is the zero vector $\mathbf{0}$, then this won't work; however, in that case, $\|\mathbf{u} \times \mathbf{v}\| = 0$, so then $\mathbf{u} \times \mathbf{v}$ must be $\mathbf{0}$, which has no direction.

Like the dot product, this operation distributes over addition and associates with scalar multiplication:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w},$$
$$\mathbf{u} \times a\mathbf{v} = a(\mathbf{u} \times \mathbf{v}).$$

The latter fact is easy to see, since we have a corresponding fact for $\|\mathbf{u} \times a\mathbf{v}\|$ and the direction of $\mathbf{u} \times a\mathbf{v}$ reverses when a is negative. The first of these is more difficult; it uses the result for $\|\mathbf{u} \times \mathbf{v}\|$ in terms of $\operatorname{proj}_{\mathbf{u}}^{\perp} \mathbf{v}$. This allows you to draw everything in the plane perpendicular to $\mathbf{u}$; if you look in the direction of $\mathbf{u}$ when looking at this plane, then $\mathbf{u} \times \mathbf{v}$ rotates $\operatorname{proj}_{\mathbf{u}}^{\perp} \mathbf{v}$ (which is in this plane) clockwise through a right angle and scales it by $\|\mathbf{v}\|$; since both this operation and projection distribute over addition, so does the cross product itself.

However, there is one important difference between the properties of the dot and cross products:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

This is because, while the magnitudes are the same, the directions are reversed, since you're curling your fingers the other way.

For practical calculations, it's again enough to know what happens to the standard basis vectors:

$$\mathbf{i} \times \mathbf{i} = \mathbf{0},\ \mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{i} \times \mathbf{k} = -\mathbf{j},$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k},\ \mathbf{j} \times \mathbf{j} = \mathbf{0},\ \mathbf{j} \times \mathbf{k} = \mathbf{i},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j},\ \mathbf{k} \times \mathbf{j} = -\mathbf{i},\ \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

Based on this,

$$\begin{aligned}\langle a, b, c\rangle \times \langle d, e, f\rangle &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = (bf - ce)\mathbf{i} + (cd - af)\mathbf{j} + (ae - bd)\mathbf{k} \\ &= \langle bf - ce, cd - af, ae - bd\rangle.\end{aligned}$$

For example,

$\langle 1, -2, 0\rangle \times \langle 2, 2, 1\rangle = \langle(-2)(1) - (0)(2), (0)(2) - (1)(1), (1)(2) - (-2)(2)\rangle = \langle -2 - 0, 0 - 1, 2 + 4\rangle = \langle -2, -1, 6\rangle.$

If you know about determinants, then you can think of

$$\langle a, b, c\rangle \times \langle d, e, f\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix};$$

the value of this determinant is the value of the cross product above.

Along with the cross product, people often look at the so-called *triple scalar product* of three vectors in $\mathbf{R}^3$; this is simply

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

This can be calculated with determinants as well; then the top row of the determinant, instead of consisting of the standard basis vectors, now consists of the components of $\mathbf{u}$ to go with the components of $\mathbf{v}$ and $\mathbf{w}$ in the other rows. Geometrically, this represents a volume; more precisely, $|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}|$ is the volume of a parallelepiped whose edges are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ is positive if you can curl the fingers of your right hand from $\mathbf{u}$ to $\mathbf{v}$ and stick out your thumb along $\mathbf{w}$ but negative if your thumb points the wrong way.

The cross product in two dimensions

For vectors in $\mathbf{R}^2$, we can interpret $\mathbf{u} \times \mathbf{v}$ as a scalar, so this is sometimes callled the *scalar cross product*. The absolute value $|\mathbf{u} \times \mathbf{v}|$ is the $\|\mathbf{u} \times \mathbf{v}\|$ from above; $\mathbf{u} \times \mathbf{v}$ itself is positive if you turn counterclockwise to go from $\mathbf{u}$ to $\mathbf{v}$ but negative if you turn clockwise. (Here I'm assuming a counterclockwise coordinate system: the rotation from the positive x-axis to the positive y-axis is counterclockwise.) If $\mathbf{u}$ and $\mathbf{v}$ are parallel (or antiparallel), or if either of them is the zero vector $\mathbf{0}$, then $\mathbf{u} \times \mathbf{v}$ is just 0.

The cross product in 2 dimensions follows the same algebraic rules as in 3 dimensions:

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}, \\ \mathbf{u} \times a\mathbf{v} &= a(\mathbf{u} \times \mathbf{v}), \\ \mathbf{u} \times \mathbf{v} &= -\mathbf{v} \times \mathbf{u}. \end{aligned}$$

If anything, these are easier to establish geometrically than the corresponding properties in $\mathbf{R}^3$.

Another way to think of the scalar cross product is to embed $\mathbf{R}^2$ within $\mathbf{R}^3$; that is, we take the z-coordinate of every point to be fixed (typically $z = 0$), so that the z-component of every vector is $\Delta z = 0$. Then instead of the scalar cross product $\mathbf{u} \times \mathbf{v}$, you can speak of the triple scalar product $\mathbf{k} \cdot \mathbf{u} \times \mathbf{v}$. Yet another way to think of it is as a dot product; much as $a - b$ is the sum of a and $-b$, so $\mathbf{u} \times \mathbf{v}$ is the dot product of $\mathbf{u}$ and $\times\mathbf{v}$, where $\times\mathbf{v}$ is simply $\mathbf{v}$ rotated clockwise through a right angle. You can also speak of signed angles in 2 dimensions; if you treat a counterclockwise angle as positive and a clockwise angle as negative, then

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin \bar{\angle}(\mathbf{u}, \mathbf{v}),$$

where the bar over the angle symbol indicates a signed angle.

For practical calculations, since $\mathbf{i} \times \mathbf{i} = 0$, $\mathbf{i} \times \mathbf{j} = 1$, $\mathbf{j} \times \mathbf{i} = -1$, and $\mathbf{j} \times \mathbf{j} = 0$, the formula is

$$\langle a, b \rangle \times \langle c, d \rangle = ad - bc.$$

For example,

$$\langle 1, -2 \rangle \times \langle 3, 5 \rangle = (1)(5) - (-2)(3) = 5 + 6 = 11.$$

If you know about determinants, then

$$\langle a, b \rangle \times \langle c, d \rangle = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Similarly,

$$\times \langle a, b \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ a & b \end{vmatrix} = \langle b, -a \rangle.$$

Cross products in more than 3 dimensions can also be done, but in that case the result is neither a scalar nor a vector but a more general concept called a *tensor*. We will not be using these.

Orientation

The dot and cross products both rely on the geometric notion of length, but the cross product additionally depends on an **orientation**; this is the choice between the right-hand and left-hand rules (in 3 dimensions) or between counterclockwise and clockwise angles (in 2 dimensions). While our physical space really does have lengths and angles, the choice of orientation is arbitrary, so results that apply to geometry shouldn't depend on it.

Just as we can distinguish row vectors from column vectors in situations where lengths don't make sense, so we can distinguish axial vectors from polar vectors in situations where orientation is arbitrary. So, a **polar vector** is an ordinary vector representing a change in position, but an **axial vector** or **pseudovector** is a vector together with a choice of orientation, where we may reverse our choice of orientation as we please so long as we replace the vector with its opposite when we do so. For example, while a polar vector in $\mathbf{R}^3$ may be fully described as $\langle -2, -1, 6\rangle$, an axial vector in $\mathbf{R}^3$ might be described as $\langle -2, -1, 6\rangle$ right-handed, or (for the *same* axial vector) as $\langle 2, 1, -6\rangle$ left-handed. Thus you can say, for example,

$$\langle 1, -2, 0\rangle \times \langle 2, 2, 1\rangle = \langle -2, -1, 6\rangle \text{ right-handed} = \langle 2, 1, -6\rangle \text{ left-handed}.$$

Similarly, a **pseudoscalar** is a scalar together with a choice of orientation, where again we may reverse our choice of orientation as we please so long as we replace the scalar with its opposite. In $\mathbf{R}^2$, the cross product of two vectors is a pseudoscalar; in $\mathbf{R}^3$, the triple scalar product of three vectors is a pseudoscalar. For example,

$$\langle 1, -2\rangle \times \langle 3, 5\rangle = 11 \text{ counterclockwise} = -11 \text{ clockwise},$$

and

$$\langle 1, -2, 0\rangle \cdot \langle 2, 2, 1\rangle \times \langle 0, 3, 5\rangle = 27 \text{ right-handed} = -27 \text{ left-handed}.$$

Axial vectors obey the same rules of arithmetic as polar vectors; here is a list of operations with these that make sense in $\mathbf{R}^3$:

- Addition: adding a polar vector to a point to get another point, adding two polar vectors together to get another polar vector, adding two axial vectors together to get another axial vector;
- Subtraction: subtracting a polar vector from a point to get another point, subtracting one polar vector from another to get another polar vector, subtracting one axial vector from another to get another axial vector;
- Scalar multiplication: multiplying a polar vector by a scalar to get another polar vector, multiplying an axial vector by a scalar to get another axial vector, multiplying a polar vector by a pseudoscalar to get an axial vector, multiplying an axial vector by a pseudoscalar to get a polar vector;
- Inner multiplication (dot product): multiplying two polar vectors to get a scalar, multiplying a polar vector and an axial vector to get a pseudoscalar, multiplying two axial vectors to get a scalar;
- Outer multiplication (cross product): multiplying two polar vectors to get an axial vector, multiplying a polar vector and an axial vector to get a polar vector, multiplying two axial vectors to get an axial vector.

Similarly, pseudoscalars can be added or subtracted to produce more pseudoscalars and can be multiplied together to produce an ordinary scalar, or you can multiply a scalar and a pseudoscalar to produce another pseudoscalar. In $\mathbf{R}^2$, the list of operations is the same, except that the result of a cross product is a scalar or a pseudoscalar rather than a vector (a polar vector) or a pseudovector (an axial vector).

The rule of thumb for all of this is that you can only add or subtract things that are alike in every way, but you can multiply anything together; the result is 'pseudo' if you multiplied together an odd number of pseudothings (so pseudos cancel, like minus signs, in pairs), where the cross product introduces an extra pseudo.

In the most general case, where you don't have a good notion of length and also don't have any way to prefer one orientation over another, you have polar column vectors (the ordinary notion of vector), axial column vectors, polar row vectors, and axial row vectors. In general, only polar column vectors can interact with points. None of this affects calculations when properly done, but like keeping track of units, keeping track of these can prevent you from accidentally doing meaningless calculations.

Linear geometry

Here I'll summarize the formulas in Section 11.5 of the textbook that can be made simpler by doing arithmetic with points and vectors (instead of just with vectors as the book does) or by using the two-dimensional cross product (instead of only the three-dimensional cross product as the book does).

A parametric equation for the line through a point P_0 in the direction of a nonzero vector $\mathbf{v}$ is

$$P = P_0 + t\mathbf{v},$$

where t is the parameter and $P = (x, y)$ or $P = (x, y, z)$ is a point on the line. Similarly, a parametric equation for the line through points P_1 and P_2 is

$$P = P_1 + t(P_2 - P_1).$$

A nonparametric equation for the line through P_0 in the direction of $\mathbf{v}$ in 2 dimensions is

$$(P - P_0) \times \mathbf{v} = 0.$$

Similarly, a system of equations for the line through P_0 in the direction of $\mathbf{v}$ in 3 dimensions is

$$(P - P_0) \times \mathbf{v} = \mathbf{0}.$$

(The only difference is whether the zero on the right-hand side is the scalar 0 or the vector $\mathbf{0}$.)

The distance from a point S to the line through P_0 in the direction of $\mathbf{v}$ is

$$\|(S - P_0) \times \hat{\mathbf{v}}\| = \frac{\|(S - P_0) \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

Similarly, the distance from S to the line through P_1 and P_2 is

$$\frac{\|(S - P_1) \times (P_2 - P_1)\|}{\|P_2 - P_1\|}.$$

An equation for the line (in 2 dimensions) or plane (in 3 dimensions) through P_0 and perpendicular to a vector $\mathbf{n}$ is

$$(P - P_0) \cdot \mathbf{n} = 0.$$

Finally, the distance from S to the line or plane through P_0 and perpendicular to $\mathbf{n}$ is

$$\|(S - P_0) \cdot \hat{\mathbf{n}}\| = \frac{\|(S - P_0) \cdot \mathbf{n}\|}{\|\mathbf{n}\|}.$$